Lecture 4. *Last updated 03.02.10*

IV. COVARIANT DIFFERENTIATION

A. Parallel translation

In Special Relativity if A_i is a vector dA^i is also a vector (the same is valid for any tensor). But in curvilinear coordinates this is not the case:

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k \tag{IV.1}$$

$$dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k \frac{\partial^2 x'^k}{\partial x^i \partial x^l} dx^l, \tag{IV.2}$$

thus dA_i is not a vector unless x'^k are linear functions of x^k (like in the case of Lorentz transformations). Let us introduce the following very useful notation:

$$, i = \frac{\partial}{\partial x^i} \tag{IV.3}$$

According to the principle of covariance we can not afford to have not tensors in any physical equations, thus we should replace all differentials like

$$dA_i \text{ and } \frac{\partial A_i}{\partial x^k} \equiv A_{i,k} \tag{IV.4}$$

by some corrected values which we will denote as

$$DA_i \text{ and } A_{i;k} \tag{IV.5}$$

correspondingly. In arbitrary coordinates to obtain a differential of a vector which forms a vector we should subtract vectors in the same point, not in different as we have done before.

Hence, we need produce a parallel transport or a parallel translation. Under a parallel translation of a vector in galilean frame of reference its components don't change, but in curvilinear coordinates they do and we should introduce some corrections:

$$DA^i = dA^i - \delta A^i. \tag{IV.6}$$

These corrections obviously should be linear with respect to all components of A_i and independently they should be linear with respect of dx^k, hence we can write these corrections as

$$\delta A^i = -\Gamma^i_{kl} A^k dx^l, \tag{IV.7}$$

where Γ^i_{kl} are called Christoffel Symbols which obviously don't form any tensor, because DA_i is the tensor while as we know dA_i is not a tensor.

B. Covariant derivatives and Christoffel symbols

In terms of the Christoffel symbols

$$DA^i = (\frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k) dx^l = (A^i_{,l} + \Gamma^i_{kl} A^k) dx^l, \tag{IV.8}$$

$$DA_i = (\frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k) dx^l = (A_{i,l} - \Gamma^k_{il} A_k) dx^l, \tag{IV.9}$$

$$A^i_{;l} = \frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k = A^i_{,l} + \Gamma^i_{kl} A^k, \tag{IV.10}$$

$$A_{i;l} = \frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k = A_{i,l} - \Gamma^k_{il} A_k. \tag{IV.11}$$

To calculate the covariant derivative of tensor let us start with contravariant tensor which can be presented as a product of two contravariant vectors $A^i B^k$. In this case the corrections under parallel transport are

$$\delta(A^i B^k) = A^i \delta B^k + B^k \delta A^i = -A^i \Gamma^k_{lm} B^l dx^m - B^k \Gamma^i_{lm} A^l dx^n, \tag{IV.12}$$

since these corrections are linear we have the same for arbitrary tensor A^{ik}:

$$\delta A^{ik} = -(A^{im}\Gamma^k_{ml} + A^{mk}\Gamma^i_{ml}) dx^l \tag{IV.13}$$

$$DA^{ik} = dA^{ik} - \delta A^{ik} \equiv A^{ik}_{;\,l} dx^l, \tag{IV.14}$$

hence

$$A^{ik}_{;\,l} = A^{ik}_{,l} + \Gamma^i_{ml} A^{mk} + \Gamma^k_{ml} A^{im} \tag{IV.15}$$

In similar way we can obtain that

$$A^i_{k;\,l} = A^i_{k,l} - \Gamma^m_{kl} A^i_m + \Gamma^i_{ml} A^m_k, \text{ and } A_{ik;\,l} = A_{ik,l} - \Gamma^m_{il} A_{mk} - \Gamma^m_{kl} A_{m,\,i}. \tag{IV.16}$$

In the most general case when we have tensor of $m+n$ rank with m contravariant and n covariant indices the rule for calculation of the covariant derivative with respect to index p is the following

$$A^{i_1\, i_2\, \dots\, i_m}_{j_1\, j_2\, \dots\, j_n\, ;\, \mathbf{p}} = A^{i_1\, i_2\, \dots\, i_m}_{j_1\, j_2\, \dots\, j_n\, ,\, \mathbf{p}} + \Gamma^{\mathbf{i_1}}_{\mathbf{kp}} A^{\mathbf{k}\, i_2\, \dots\, i_m}_{j_1\, j_2\, \dots\, j_n} + \Gamma^{\mathbf{i_2}}_{\mathbf{kp}} A^{i_1\, \mathbf{k}\, \dots\, i_m}_{j_1\, j_2\, \dots\, j_n} + \dots + \Gamma^{\mathbf{i}_m}_{\mathbf{kp}} A^{i_1\, i_2\, \dots\, \mathbf{k}}_{j_1\, j_2\, \dots\, j_n} - \tag{IV.17}$$

$$- \Gamma^{\mathbf{k}}_{\mathbf{j_1}\, \mathbf{p}} A^{i_1\, i_2\, \dots\, i_m}_{\mathbf{k}\, j_2\, \dots\, j_n} - \Gamma^{\mathbf{k}}_{\mathbf{j_2}\, \mathbf{p}} A^{i_1\, i_2\, \dots\, i_m}_{j_1\, \mathbf{k}\, \dots\, j_n} - \dots - \Gamma^{\mathbf{k}}_{\mathbf{j}_n\, \mathbf{p}} A^{i_1\, i_2\, \dots\, i_m}_{j_1\, j_2\, \dots\, \mathbf{k}}. \tag{IV.18}$$

C. The Christoffel symbols and the metric tensor

So far we don't know how the Christoffel symbols depend on coordinates, however we can prove that they are symmetric in the subscripts. Let some covariant vector A_i is the gradient of a scalar ϕ, i.e. $A_i = \phi_{,i}$. Then

$$A_{k;\,i} - A_{i;\,k} = \phi_{,k,i} - \Gamma^l_{ki}\phi_{,l} - \phi_{,i,k} + \Gamma^l_{ik}\phi_{,l} = \left(\Gamma^l_{ki} - \Gamma^l_{ik}\right)\phi_{,l}. \tag{IV.19}$$

In Galilean coordinates

$$\Gamma^l_{ik} = \Gamma^l_{ki} = 0, \quad \text{hence in Galilean coordinates } A_{k;\,i} - A_{i;\,k} = 0, \tag{IV.20}$$

but taking into account that $A_{k;\,i} - A_{i;\,k}$ is a tensor we conclude that if it equals to zero in one system of coordinates it should be equal to zero in any other coordinate system, hence

$$\Gamma^l_{ik} = \Gamma^l_{ki} \tag{IV.21}$$

in any coordinate system.
This is a typical example of the proof widely used in General Relativity:
If some equality between tensors is valid in one coordinate system then this equality is valid in arbitrary coordinate system. This is obvious advantage to deal with tensors.
Then we can show that covariant derivatives of g_{ik} are equal to zero. Indeed:

$$DA_i = g_{ik}DA^k \quad DA_i = D(g_{ik}A^k) = g_{ik}DA^k + A^kDg_{ik}, \quad \text{hence} \quad g_{ik}DA^k = g_{ik}DA^k + A^kDg_{ik}, \tag{IV.22}$$

which obviously means that

$$A^kDg_{ik} = 0. \tag{IV.23}$$

Taking into account that A^k is arbitrary vector, we conclude that

$$Dg_{ik} = 0. \tag{IV.24}$$

This is another example of proof in General Relativity: If the the sum $B_{ik}A^i = 0$ for arbitrary vector A^i then the tensor $B_{ik} = 0$. Then taking into account that

$$Dg_{ik} = g_{ik;m}dx^m = 0 \tag{IV.25}$$

for arbitrary infinitesimally small vector dx^m we have

$$g_{ik;m} = 0. \tag{IV.26}$$

Now we are ready to relate the Christoffel symbols to the metric tensor. Introducing useful notation

$$\Gamma_{k,\,il} = g_{km}\Gamma^m_{il}, \tag{IV.27}$$

we have

$$g_{ik;\,l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk}\Gamma^m_{il} - g_{im}\Gamma^m_{kl} = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,\,il} - \Gamma_{i,\,kl} = 0. \tag{IV.28}$$

Permuting the indices i, k and l twice as $i \to k$, $k \to l$, $l \to i$,we obtain

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,\,il} + \Gamma_{i,\,kl}, \quad \frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,\,kl} + \Gamma_{l,\,ik} \text{ and } -\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,\,ki} - \Gamma_{k,\,li}. \tag{IV.29}$$

Taking into account that $\Gamma_{k,\,il} = \Gamma_{k,\,li}$, after summation of these three equation we have

$$g_{ik,l} + g_{li,k} - g_{kl,i} = 2\Gamma_{i,\,kl}, \tag{IV.30}$$

and finally

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right). \tag{IV.31}$$

Now we have expressions for the Christoffel symbols in terms of the metric tensor and hence we know their dependence on coordinates.

D. Physical applications

The previous material can be summarized as follows:
Gravity is equivalent to curved space-time, hence in all differentials of tensors we should take into account the change in the components of a tensor under an infinitesimal parallel transport. Corresponding corrections are expressed in terms of the Cristoffel symbols and are reduced to replacement of any partial derivative by corresponding covariant derivative. In other words we can say that if one wants to take into account all effects of Gravity on any local physical process, described by the corresponding equations written in framework of Special Relativity, one should just replace all partial derivatives by covariant derivatives in these equation according to the following very nice and simple but actually very strong and important formulae:

$$\mathbf{d} \to \mathbf{D} \quad \text{and} \quad , \to ;. \tag{IV.32}$$

Example 1: In special Relativity

$$dg_{ik} = 0 \;\text{ and }\; g_{ik,l} = 0, \tag{IV.33}$$

while in General Relativity

$$Dg_{ik} = 0 \;\text{ and }\; g_{ik;l} = 0. \tag{IV.34}$$

Example 2: Let us apply above formulae to description of motion of a free test particle in a given gravitational field. Let

$$u^i = \frac{dx^i}{ds} \tag{IV.35}$$

is the four-velocity. Then the equation for motion of a free particle in absence of gravitational field is

$$\frac{du^i}{ds} = 0 \tag{IV.36}$$

is generalized to the equation

$$\frac{Du^i}{ds} = 0, \tag{IV.37}$$

which gives

$$\frac{Du^i}{ds} = \frac{du^i}{ds} + \Gamma^i_{kn} u^k \frac{dx^n}{ds} = \frac{d^2x^i}{ds^2} + \Gamma^i_{kn} u^k u^n = 0. \tag{IV.38}$$

Thus from physical point of view the equation

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} = 0 \tag{IV.39}$$

describes the motion of free particle in a given gravitational field and

$$\frac{d^2x^i}{ds^2} = -\Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} \tag{IV.40}$$

is the four-acceleration, while from geometrical point of view this equation is the equation for geodesics in a curved space-time. That is why all particles move with the same acceleration and now this experimental fact is not coincidence anymore but consequence of geometrical interpretation of gravity.

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