Atoms in strong laser fields

C. Figueira de Morisson Faria
Institut für theoretische Physik, Universtität Hannover
Appelstr 2, 30167 Hannover, Germany

Lecture 3: Theoretical methods II

- Saddle-point approximation
  - Standard saddle-point approximation
  - Uniform approximation

References

- G. Arfken, Mathematical Methods for Physicists (Academic Press, 1985)
- N. Bleistein and R. A. Handelsman, Asymptotic Expansions of Integrals
- R. Kopold, PhD thesis (TU-Munich, 2001)
- P. Salières et al, Science 292, 902 (2001)

Saddle-point approximation
or
Steepest descent method

Simplest case: 1D integral

$$I = \int_C dz g(z) \exp[\eta s(z)]$$

with

- z, $g(z)$ and $s(z)$ complex
- $g(z)$ and $s(z)$ analytic
- $g(z)$ should change much more slowly than $\exp[\eta s(z)]$

Main contributions: saddle points $\partial_z s(z)|_{z=z_s} = 0$

- $g(z) \simeq g(z_s)$
- $s(z) \simeq s(z_s) - 1/2(z - z_s)^2 \partial_z^2 s(z)|_{z=z_s}$

$$I \simeq \sum_{\text{saddle points}} g(z_s) \sqrt{\frac{2\pi}{\eta \partial_z^2 s(z_s)}} \exp[\eta s(z_s)]$$

Q1: Why saddle points?

Laplace equations must be satisfied!

$$\left[\partial_x^2 + \partial_y^2\right] u_{(j)}(x, y) = 0$$

with

- $x = \mathrm{Re}[z]$; $y = \mathrm{Im}[z]$
- $u_{(R)} = \mathrm{Re}[s]$; $u_{(I)} = \mathrm{Im}[s]$;

Consequence: if $\partial_x^2 u_{(j)}(x, y) > 0$ then $\partial_y^2 u_{(j)}(x, y) < 0$ or vice-versa

Q2: Why steepest descent?

- Contour chosen along $\nabla u_{(R)}$

Consequence: $u_{(I)} = const.$ since $\nabla u_{(R)}.\nabla u_{(I)} = 0$

Explanation: Cauchy-Riemann conditions

$$\begin{aligned} \partial_x u_{(R)} &= \partial_y u_{(I)} \\ \partial_y u_{(R)} &= -\partial_x u_{(I)} \end{aligned}$$

- Application: above-threshold ionization

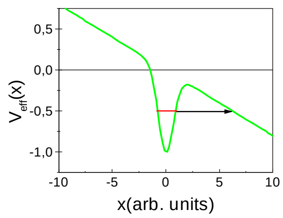
$$M_{\text{resc}} = -\int_{-\infty}^{\infty} dt \int_{-\infty}^{t} dt' \int d^3\mathbf{k}\, e^{iS_{\mathbf{p}}(t,t',\mathbf{k})} V_{\mathbf{pk}} V_{\mathbf{k}0}$$

with

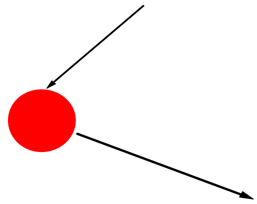
$S_{\mathbf{p}}(t,t',\mathbf{k}) = -\frac{1}{2}\int_t^{\infty} d\tau\ [\mathbf{p}-\mathbf{A}(\tau)]^2 - \frac{1}{2}\int_{t'}^{t} d\tau\ [\mathbf{k}-\mathbf{A}(\tau)]^2 + |E_0|t'$

Saddle point equations

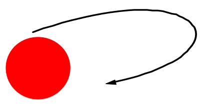
- $\partial S/\partial t' = 0 \Rightarrow [\mathbf{k}-\mathbf{A}(t')]^2 = -2|E_0|$
(energy conservation at t')



- $\partial S/\partial t = 0 \Rightarrow [\mathbf{p}-\mathbf{A}(t)]^2 = [\mathbf{k}-\mathbf{A}(t)]^2$
(energy conservation at t)

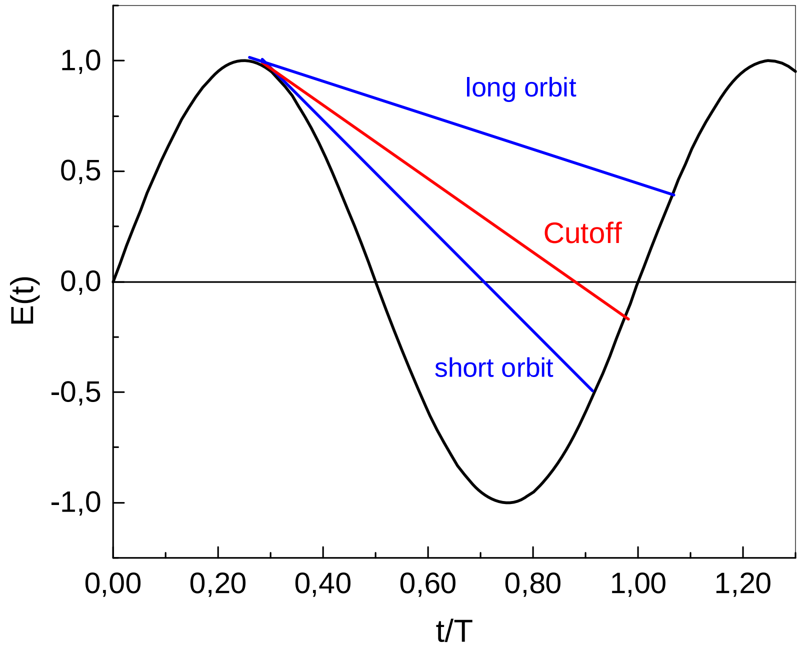


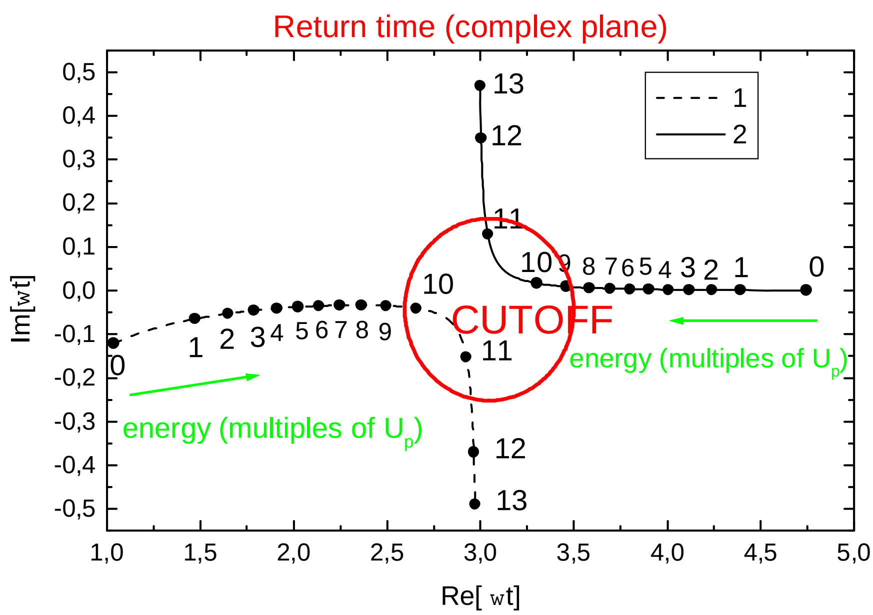
- $\partial S/\partial \mathbf{k} = 0 \Rightarrow \int_{t'}^{t} [\mathbf{k}-\mathbf{A}(\tau)]\, d\tau = 0$
(intermediate momentum)



Tunneling ionization at t': $t, t', \mathbf{k}$ complex

Saddles occurs in pairs that nearly coalesce at the cutoff





Saddle-point approximation

(M. Lewenstein et al, PRA 49, 2117 (1994); R. Kopold et al, Opt. Comm. 179, 39 (2000); P. Salières et al, Science 292, 902 (2001))

Transition amplitude

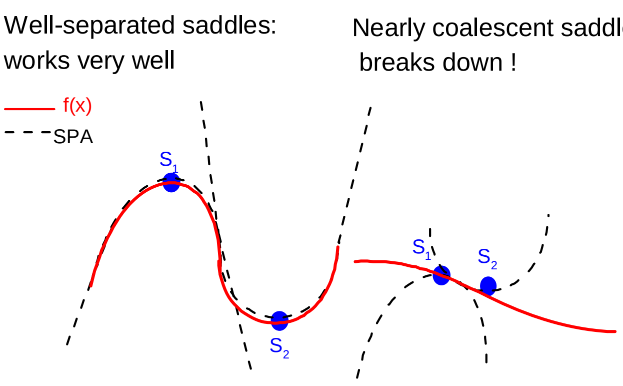
$$M_{\mathrm{resc}}^{(\mathrm{SPA})} = \sum_s \underbrace{(2\pi i)^{5/2} \frac{V_{\mathbf{p}\mathbf{k}_s} V_{\mathbf{k}_s 0}}{\sqrt{\det S''_{\mathbf{p}}(t,t',\mathbf{k})|_s}}}_{A_i} \exp\left[i S_{\mathbf{p}}(t_s, t'_s, k_s) \right]$$

- Quadratic expansion of $S_p(t,t',k)$ around $(t_s, t'_s, \mathbf{k}_s)$
- Saddles treated as independent
- One saddle discarded after the cutoff

Advantages

- Additional physical insight: space-time picture
- Simplification of computations: no multiple integrals required
- Investigation of interference processes

Drawback: breaks down for nearly coalescent saddle points!



The Uniform Approximation

- Non-uniform asymptotic expansion: becomes invalid when a parameter θ reaches a critical value θ_c
- Unform asymptotic expansion: valid over a domain containing critical values θ_c

Example (cause of a critical value): coalescent saddle points

Integral

$$I(\eta, \alpha) = \int_C dz g(z) \exp[\eta s(z; \alpha)]$$

(saddle points: $\alpha = (\alpha_+, \alpha_-)$)

Simple		Coalescent
$\partial_z s(\alpha_+, \alpha) = \partial_z s(\alpha_-, \alpha) = 0$ $\partial_z^2 s(\alpha_+, \alpha) \neq 0; \partial_z^2 s(\alpha_-, \alpha) \neq 0$ $\alpha_+ \neq \alpha_-$	$\Rightarrow$	$\partial_z s(\alpha_\pm, \alpha) = \partial_z^2 s(\alpha_\pm, \alpha) = 0$ $\partial_z^3 s(\alpha_\pm, \alpha) \neq 0$

Method

- Change of variable: $z = z(t)$
- Transformation chosen such that the number of saddle points, as well as their behavior, remains the same

Exponent

Should be as simple as possible:

$$\phi(z,\alpha) = s(z(t),\alpha) = -(t^3/3 - \gamma^2 t) + \rho$$

γ and ρ must be determined !

- $\gamma^3 = 3/4\,[s(\alpha_+,\alpha) - s(\alpha_-,\alpha)]$
- $\rho = 1/2\,[s(\alpha_+,\alpha) + s(\alpha_-,\alpha)]$

Prefactor

$$I(\eta,\alpha) = \int G_0(t,\alpha)\exp[\eta\phi(t,\gamma)]dt$$

with $G_0(t,\gamma) = g(z(t))dz/dt$

We choose

$G_0(t,\gamma) = a_0 + a_1 t + (t^2 - \gamma^2)H_0(t,\alpha)$

so that

- $a_0 = [G_0(\alpha_+,\gamma)) + G_0(\alpha_-,-\gamma)]\,/2$
- $a_1 = [G_0(\alpha_+,\gamma) - G_0(\alpha_-,-\gamma)]\,/(2\gamma)$

$$I(\eta,\alpha) \sim \exp[\eta\rho]\int(a_0 + a_1 t)\exp[-\eta(t^3/3 - \gamma^2 t)]dt + R_0$$

$$I(\eta,\alpha) \sim 2\pi i \exp[\eta\rho] \left[\frac{a_0}{\eta^{1/3}} A_i(\eta^{2/3}\gamma^2) + \frac{a_1}{\eta^{2/3}} A_i'(\eta^{2/3}\gamma^2) \right]$$

Using

- $A_i(z) = \frac{1}{3}\sqrt{z}\left[J_{1/3}(2/3z^{3/2}) + J_{-1/3}(2/3z^{3/2})\right]$
- $A_i'(z) = \frac{1}{3}z\left[J_{-2/3}(2/3(-z)^{3/2}) - J_{2/3}(2/3z^{3/2})\right]$

and

- $A_1 = \sqrt{\frac{2\pi}{\eta}} g(\gamma,\alpha_-)/\sqrt{s''(\alpha_-,\alpha)}$
- $A_2 = \sqrt{\frac{2\pi}{\eta}} g(\gamma,\alpha_+)/\sqrt{s''(\alpha_+,\alpha)}$

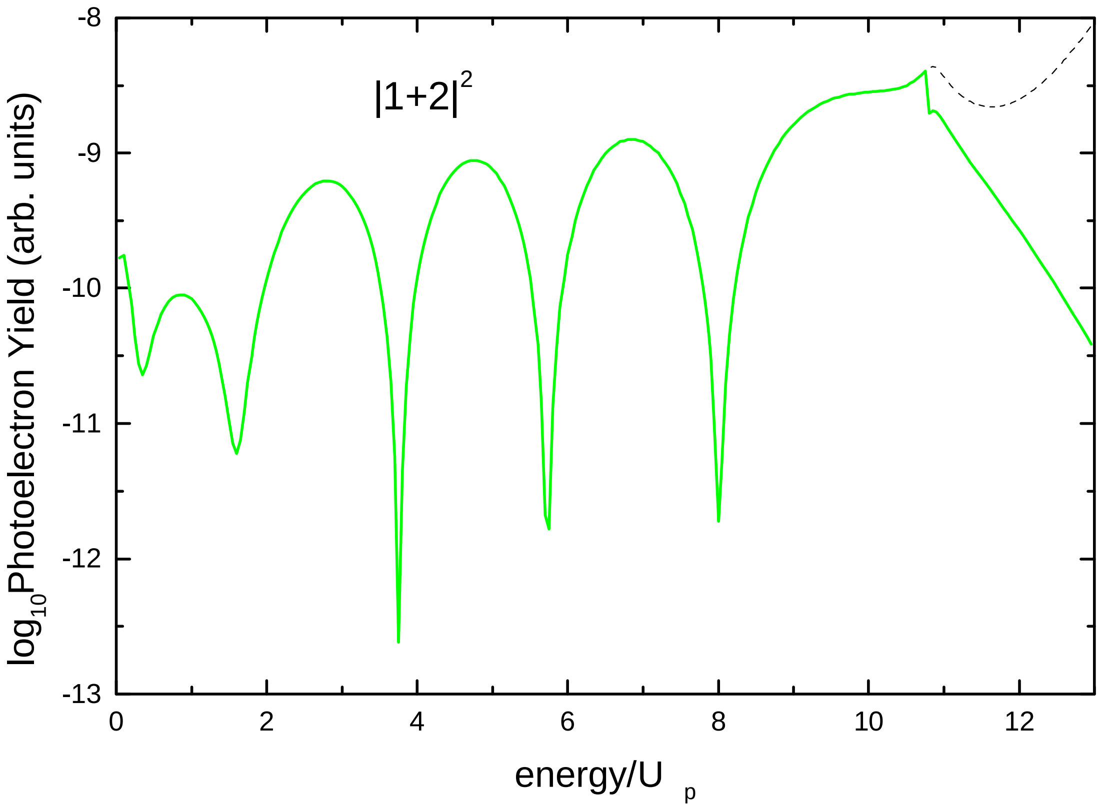
we have

$$\begin{aligned} M_{1+2} \simeq\ & \sqrt{2\pi i\eta\Delta S/3}\exp[\eta\overline{S}]\exp[i\pi/4] \times \\ & \left\{ \left(\frac{iA_1 - A_2}{2}\right)\left[J_{1/3}(-i\eta\Delta S) + J_{-1/3}(-i\eta\Delta S)\right] + \right. \\ & \left. \left(\frac{iA_2 - A_1}{2}\right)\left[J_{-2/3}(-i\eta\Delta S) - J_{2/3}(-i\eta\Delta S)\right] \right\} \end{aligned}$$

Standard SPA: recovered using

$$J_{\pm\nu}(z) \sim \left(\frac{2}{\pi z}\right)\cos(z \mp \nu - \frac{\pi}{4})$$

(valid for large z)



|1+2|^2
log$_{10}$Photoelectron Yield (arb. units)
energy/U$_p$
-8
-9
-10
-11
-12
-13
0
2
4
6
8
10
12

Transition amplitude

- Classically allowed region

$$M_{i+j} = \sqrt{2\pi\Delta S/3}\,\exp(i\bar{S} + i\pi/4)\Big\{\bar{A}[J_{1/3}(\Delta S){+}J_{-1/3}(\Delta S)]$$
$$+\Delta A[J_{2/3}(\Delta S) - J_{-2/3}(\Delta S)]\Big\}$$

- Classically forbidden region

$$M_{i+j} = \sqrt{2i\Delta S/\pi}\,\exp(i\bar{S})\,\Big[\bar{A}K_{1/3}(-i\Delta S)$$
$$+i\Delta A K_{2/3}(-i\Delta S)\Big]$$

- with

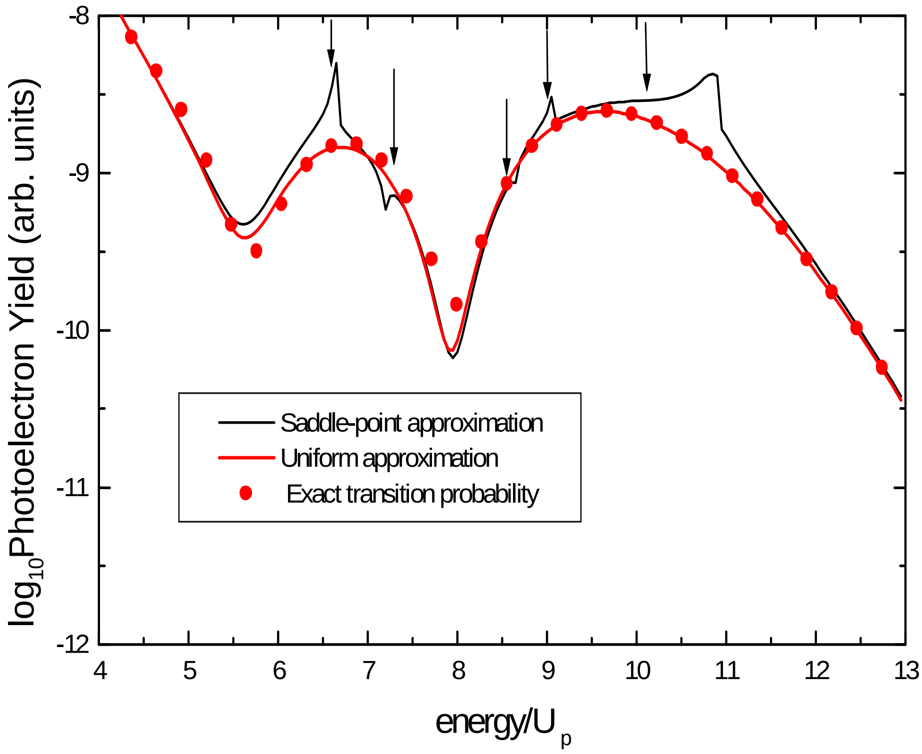
$$\Delta S = (S_i - S_j)/2,$$
$$\bar{S} = (S_i + S_j)/2,$$
$$\Delta A = (A_i - iA_j)/2,$$
$$\bar{A} = (iA_i - A_j)/2,$$

The collective contribution of a pair of saddle points is considered

- Cubic expansion around the saddle points
- Valid in all energy regions
- No more complicated than the standard SPA
- Well-separated saddles: the former approximation is recovered

Application

Spectra (zero-range potential)



All cusps disappear
Results practically identical to the "exact" ones!

The angular distributions and the influence of the binding potential have also been investigated (PRA 66, 043413 (2002))