

IV - Coherence

* Key ideas

- EVERY electromagnetic field has fluctuations. Real light sources, for instance, have a spatial extension, undergo irregular fluctuations in intensity, phase, etc.
- The existence of such fluctuations can be inferred from experiments probing correlations between them at two or more space-time points (interference experiments)
- Optical coherence theory: studies optical correlation phenomena / the statistical description of optical fields
=> Precise measure of the correlation between fluctuating fields

* Definition: Let us consider two sources, emitting the fields

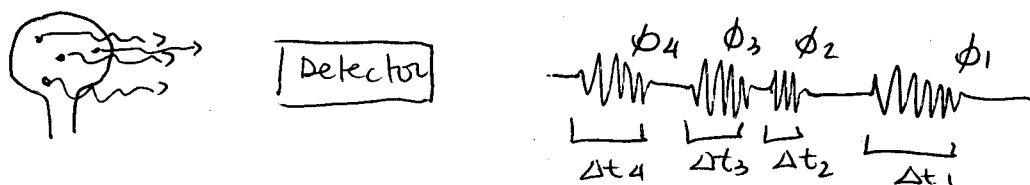
$$E_1 = E_{01} \exp[i(\vec{k}_1 \cdot \vec{r} - \omega t + \phi_1)] + c.c.$$

$$E_2 = E_{01} \exp[i(\vec{k}_2 \cdot \vec{r} - \omega t + \phi_2)] + c.c.$$

$\phi_1 - \phi_2 = \text{const} \Rightarrow$ the sources are mutually coherent

1 - Temporal coherence

Let us consider a quasi monochromatic source, emitting randomly phased wave trains



• Finite bandwidth of radiation $\Delta \nu$ exists

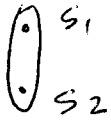
Fringes will be formed if $\Delta\theta \Delta s \leq \bar{\lambda}$

(4)

$\bar{\lambda}$
↓
mean wavelength of the light

Explanation: Each source point S_1, S_2 gives rise to a different pattern in P , whose maxima/minima will be displaced with respect to each other

As a increases, these patterns will get more and more out of step and the fringes will disappear

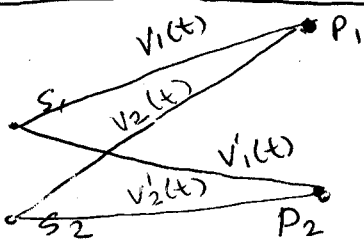


Simplified model: S_1, S_2 : two point sources (quasi monochromatic)

$\bar{\nu}$: mean frequency (the same)

$\Delta\nu$: effective spectral range (the same)

The sources are statistically independent



$V_i (i=1,2)$: signals reaching P_1

$V_i' (i=1,2)$: signals reaching P_2

We will assume that the differences between

$\overline{S_1 P_1}$ and $\overline{S_1 P_2}$

$\overline{S_2 P_1}$ and $\overline{S_2 P_2}$

are small compared to the coherence length

$$\Rightarrow V_2'(t) = V_i(t) \quad (i=1,2) \quad (*)$$

(*) Total field at P_1 :

$$V(P_1, t) = V_1(t) + V_2(t)$$

(*) Total field at P_2 :

$$V(P_2, t) = V_1'(t) + V_2'(t)$$

V_1 and V_2 are not correlated. Due to (*), however, their sum will.

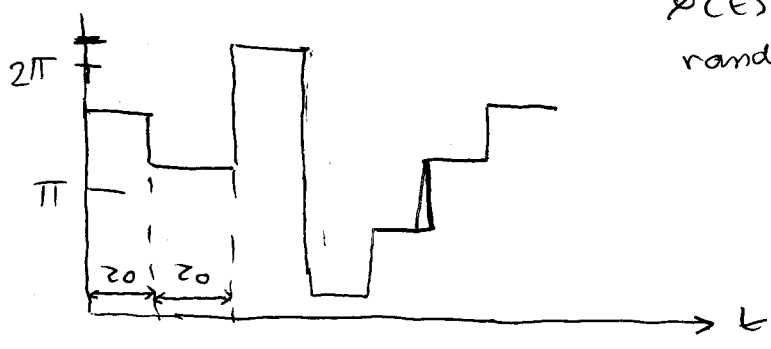
• Observed intensity, i.e., the summed radiation, will vary slowly in frequency and amplitude

Phase-locked radiation will exist for a time z_0 ,

$z_0 = \langle \Delta t \rangle = \frac{1}{\Delta \nu}$ is the coherence time of this light
 \equiv average time for which this light is coherent

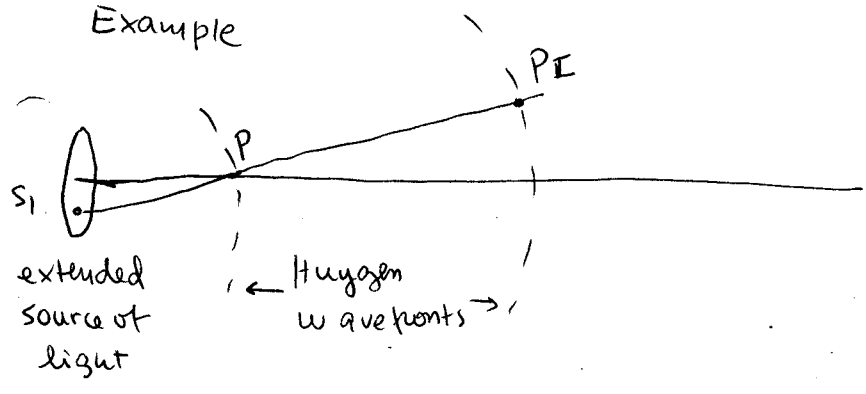
$$E(t) = \text{Re} [E_0 \exp \{-i\omega t + i\phi(t)\}]$$

$\phi(t)$ is a phase which changes randomly



Coherence length: length over which light is coherent = l_c

Example



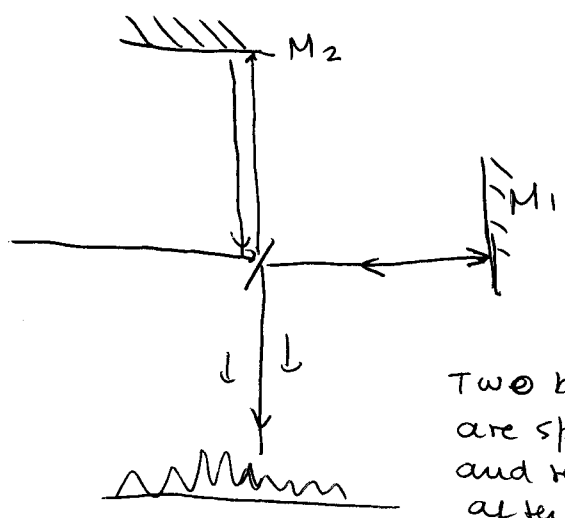
$l_c \gg PPI$: the events at P are highly correlated with those at P_I

$l_c \ll PPI$: the events at P are uncorrelated with those at P_I

$l_c = z_0 c$
↑ coherence time
↓ coherence length

$$\Delta l_c = \frac{c}{\Delta \nu}$$

Example : Michelson interferometer



Fringes occur if

$$\Delta t \Delta \nu \lesssim 1$$

$$\Rightarrow \Delta t \approx \frac{1}{\Delta \nu} \text{ coherence time}$$

$$\Delta l \approx \frac{c}{\Delta \nu} \text{ coherence length}$$

Two beams are split and reunited after a path difference $\Delta l = c \Delta t$

Physical explanation: Each frequency component will form a pattern with a different periodicity

\Rightarrow With increasing time delay, their addition will lead to a less and less well-defined fringe pattern, until no pattern will be formed

Estimates (z_c and l_c)

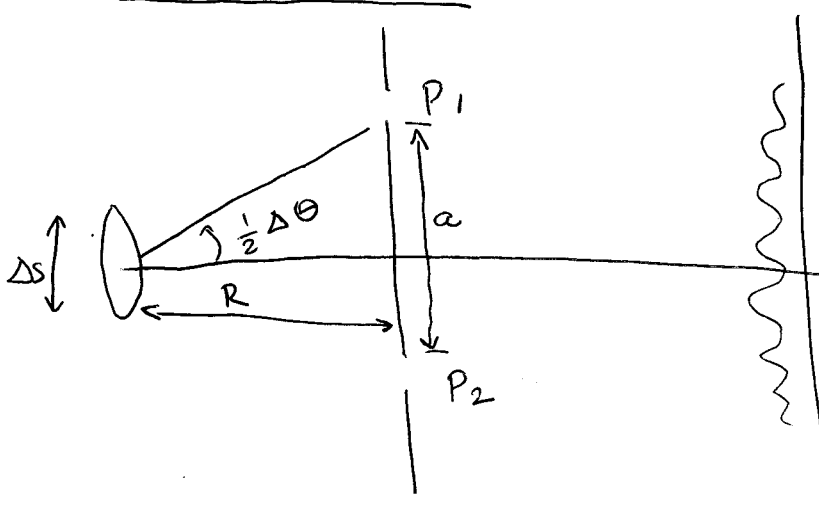
• Incandescent light: $\Delta \nu \approx 10^8 \text{ s}^{-1} \Rightarrow z_c \approx 10^{-8} \text{ s}$

$$\Delta l_c \approx 3 \times 10^8 \text{ ms}^{-1} \times 10^{-8} \text{ s} = 3 \text{ m}$$

• Laser light: $\Delta \nu \approx 10^4 \text{ s}^{-1} \Rightarrow z_c \approx 10^{-4} \text{ s}$

$$l_c \approx 30 \text{ km}$$

2 - Spatial coherence



Young's double slit experiment

The existence of fringes will depend on the separation a .

These patterns are a manifestation of spatial coherence between the two light beams reaching P from P_1, P_2 .

$P \quad P_1, P_2$

3 - Classical theory of partial coherence

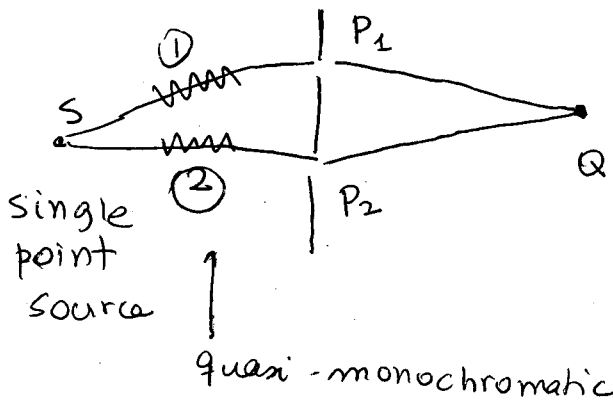
3. (a) - Temporal coherence

Coherence between two fields arriving at the same point in space through different optical paths

① Fringe visibility:

(*) Starting point: Let us consider once more Young's double-slit experiment, but now assume that the radiation is only partly coherent.

(*) Goal: Examine the degree of coherence in the radiation field using P_1 and P_2



Partial coherence:
 We cannot write the resulting field as

$$\vec{E} = E_1 \sin(\omega t + \phi_1) + E_2 \sin(\omega t + \phi_2)$$

At Q we observe a time-averaged superposition of the events coming from P_1, P_2 along paths 1 and 2

(*) Observed intensity at Q:

$$I = \langle \vec{E} \cdot \vec{E}^* \rangle_t \quad \text{where} \quad \langle f(t) \rangle_t = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t') dt'$$

(temporal average)

We will consider

$$\vec{E} = k_1 \vec{E}_1 + k_2 \vec{E}_2$$

k_1 and k_2 are propagators and depend on ① phase shifts
 ② amplitude reduction
 (differences could be caused e.g. by lenses, absorbing)

materials, etc.)

$k_1 = k_2 = 1 \Rightarrow$ the media along paths ① and ② are identical

$$I = \langle (\vec{E}_1 + \vec{E}_2) \cdot (\vec{E}_1^* + \vec{E}_2^*) \rangle_t$$

$$= \langle |\vec{E}_1|^2 + |\vec{E}_2|^2 + 2 \operatorname{Re} \langle \vec{E}_1 \cdot \vec{E}_2^* \rangle \rangle_t \quad (4)$$

⊕ Simplifications:

- All quantities are stationary (the time average is independent of the choice of the origin of time)
- The optical fields have the same polarization, so that their vectorial nature can be ignored.

$$(4) \Rightarrow \underbrace{\langle |\vec{E}_1|^2 \rangle}_{I_1} + \underbrace{\langle |\vec{E}_2|^2 \rangle}_{I_2} + \underbrace{2 \operatorname{Re} \langle \vec{E}_1 \cdot \vec{E}_2^* \rangle}_{\text{interference term (or coherence term)}} = I$$

Assumptions : the time taken by path ① is t
 the time taken by path ② is $t + \tau$

Then the interference term is $2 \operatorname{Re} \langle \vec{E}_1(t) \vec{E}_2^*(t + \tau) \rangle = 2 \operatorname{Re} \Gamma_{12}(\tau)$

$\Gamma_{12}(\tau) \equiv$ mutual coherence function or correlation function of two fields E_1 and E_2

Similarly, the functions

$$\Gamma_{11}(\tau) = \langle E_1(t) E_1^*(t + \tau) \rangle$$

$$\Gamma_{22}(\tau) = \langle E_2(t) E_2^*(t + \tau) \rangle$$

are known as self-coherence functions or autocorrelation functions

Note that $\Gamma_{11}(0) = I_1$
 $\Gamma_{22}(0) = I_2$

It is sometimes convenient to use the degree of partial coherence

(7)

$$\gamma_{12}(z) = \frac{\Gamma_{12}(z)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}} = \frac{\Gamma_{12}(z)}{\sqrt{I_1 I_2}}$$

$$\therefore I = I_1 + I_2 + 2\sqrt{I_1 I_2} \operatorname{Re} \gamma_{12}(z)$$

$\gamma_{12}(z)$ is a complex periodic function of z
 \Rightarrow an interference pattern occurs if $|\gamma_{12}(z)| \neq 0$

(a) $|\gamma_{12}| = 1 \Rightarrow$ complete coherence

(b) $0 < |\gamma_{12}| < 1 \Rightarrow$ partial coherence

(c) $|\gamma_{12}| = 0 \Rightarrow$ complete incoherence

The intensity of the fringes will vary between

$$I_{\max} = I_1 + I_2 + 2\sqrt{I_1 I_2} |\gamma_{12}|$$

$$I_{\min} = I_1 + I_2 - 2\sqrt{I_1 I_2} |\gamma_{12}|$$

$$\text{Fringe visibility } V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{2\sqrt{I_1 I_2} |\gamma_{12}|}{I_1 + I_2}$$

$$I_1 = I_2 \Rightarrow V = \frac{2\sqrt{I_1^2} |\gamma_{12}|}{2I_1} = |\gamma_{12}|$$

• Complete coherence: $|\gamma_{12}| = 1 \Rightarrow$ the interference fringes have the maximum contrast of unity

• Complete incoherence: $|\gamma_{12}| = 0 \Rightarrow$ there are no fringes

② Self coherence of quasi monochromatic light

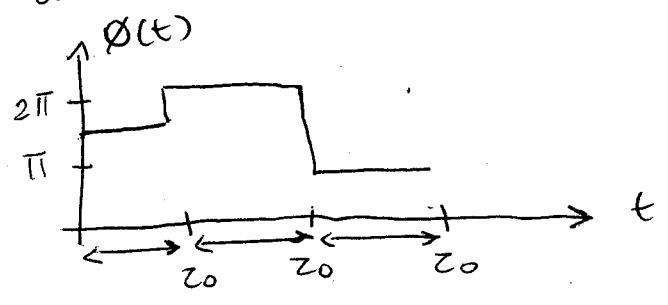
In $\gamma_{12}(z)$, $z \equiv$ time difference between events on paths ① and ②

In $\gamma_{11}(z)$, $z \equiv$ time difference between two events on the same path ①.

Let us now assume that:

① A beam of light is divided into 2 beams to produce interference $\Rightarrow |E_1| = |E_2| = |E|$

② Quasi-monochromatic light: $E(t) = E_0 e^{-i\omega t} e^{i\phi(t)}$, where ϕ is a random step function



we wish to compute the degree of self-coherence of this beam

$$\gamma_{11}(z) = \gamma_{22}(z) = \gamma(z)$$

$$\begin{aligned} \gamma(z) &= \frac{\langle E(t) E^*(t+z) \rangle}{[\langle E(t) E^*(t) \rangle \langle E(t) E^*(t) \rangle]^{1/2}} \\ &= \frac{\langle E(t) E^*(t+z) \rangle}{\langle E(t) E^*(t) \rangle} \end{aligned}$$

$$\gamma(z) = \frac{\langle E_0^2 \exp[i(\phi(t) - \omega t)] \exp[-i(\phi(t+z) - \omega(t+z))] \rangle}{E_0^2}$$

$$\gamma(z) = \exp(i\omega z) \langle \exp[i\phi(t) - \phi(t+z)] \rangle$$

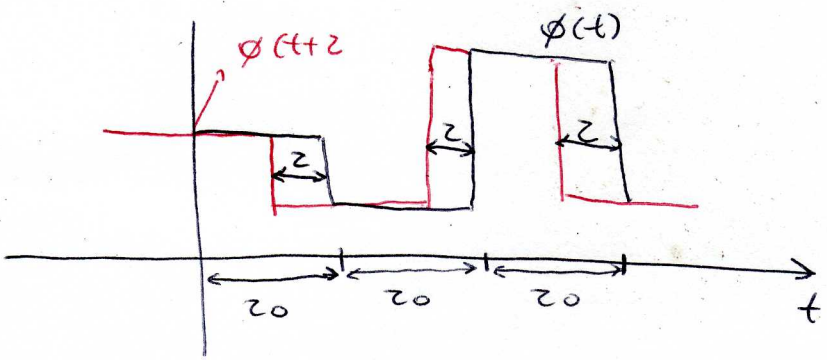
$$= \exp[i\omega z] \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \exp[i(\phi(t) - \phi(t+z))] dt$$

$\tau > z_0 \Rightarrow$ The relative phases are completely random and the time average vanishes

$$\gamma = 0$$

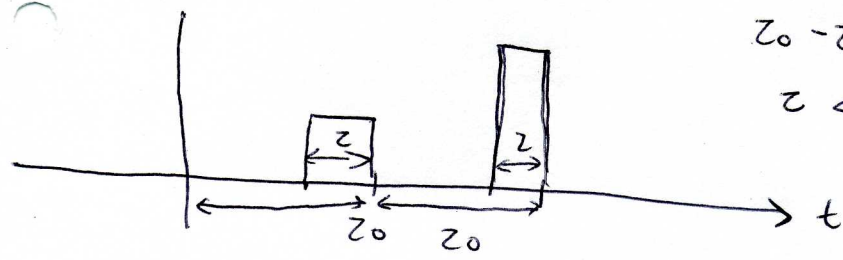
$$z < z_0$$

Let us have a closer look at the behavior of $\phi(t) - \phi(t+z)$



$$\phi(t) - \phi(t+z) = \Delta\phi$$

$0 < t < z_0 - z \Rightarrow \Delta\phi = 0$
 $z_0 - z < t < z_0 \Rightarrow \Delta\phi = \Delta$ (random value)
 $z < t < 2z_0 - z \Rightarrow \Delta\phi = 0$



Temporal average:

$$\frac{1}{\text{number of intervals}} \left[\frac{1}{z_0} \int_0^{z_0-z} dt + \frac{1}{z_0} \int_{z_0-z}^{z_0} e^{i\Delta} dt + \dots \right] = \frac{z_0-z}{z_0} + \frac{z}{z_0} e^{i\Delta} + \dots$$

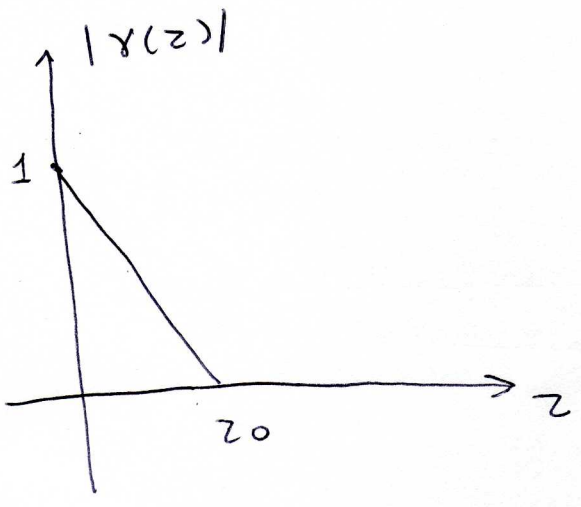
$\underbrace{\hspace{10em}}_{z_0 - z}$
 $\underbrace{\hspace{10em}}_{z e^{i\Delta}}$

$e^{i\Delta}$ will average to zero over many intervals, as Δ

is random, while the other term is the same for all intervals

$$\chi(z) = \begin{cases} (1 - \frac{z}{z_0}) e^{i\omega z} & z < z_0 \\ 0 & z \geq z_0 \end{cases}$$

$$|\chi(z)| = \begin{cases} 1 - \frac{z}{z_0} & z < z_0 \\ 0 & z \geq z_0 \end{cases}$$



The fringe visibility drops to zero if $z > z_0$.

Coherence length:

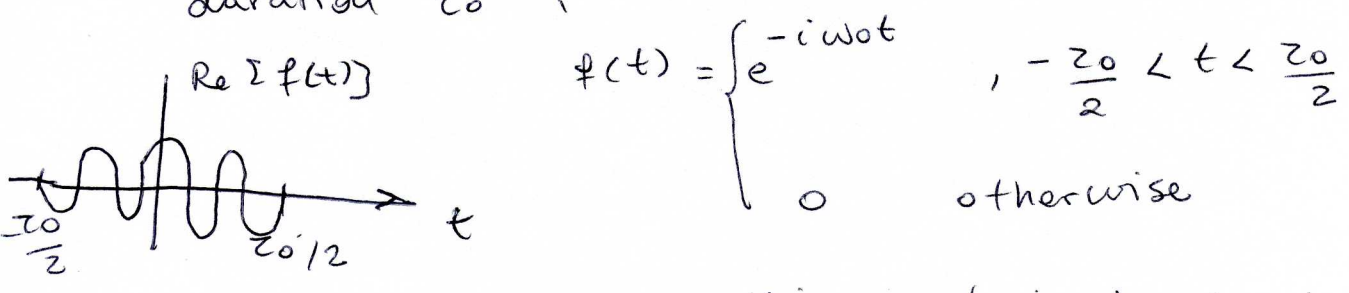
$$c z_0 = l c$$

(in this particular case, it is the length of an uninterrupted wave train)

③ Spectral resolution of a finite wave train

⊛ Aim: Investigate the relationship between the frequency spread and the coherence of a light source.

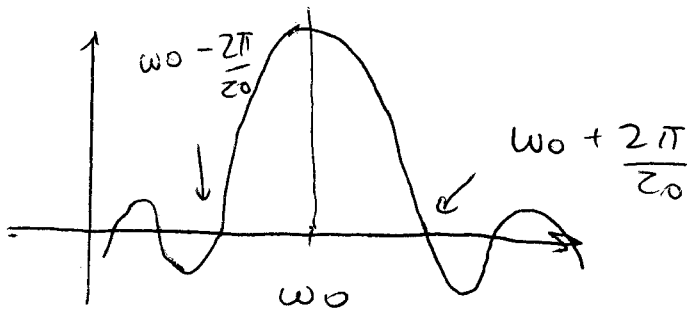
Let us consider a single wave train of finite duration z_0 .



The frequency spread of this wave train is given by the Fourier transform

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-z_0/2}^{z_0/2} e^{i(\omega - \omega_0)t} dt$$

$$= \sqrt{\frac{z_0}{\pi}} \frac{\sin[(\omega - \omega_0)z_0/2]}{\omega - \omega_0}$$



Most of the energy is contained in the region between the 1st 2 minima

$$\Rightarrow \Delta\omega = \frac{2\pi}{z_0} \Rightarrow \Delta\nu = \frac{1}{z_0}$$

In a real source, $z_0 \rightarrow \langle z_0 \rangle$

$$\langle z_0 \rangle = \frac{1}{\Delta\nu} \quad l_c = c \langle z_0 \rangle = \frac{c}{\Delta\nu}$$

$$c = \nu \lambda$$

$$\nu = \frac{c}{\lambda} \Rightarrow \Delta\nu = \frac{\Delta\lambda}{\lambda^2} \Rightarrow \boxed{l_c = \frac{\lambda^2}{\Delta\lambda}}$$

Examples:

① Discharge tubes:

$$\Delta\lambda = 0.1 \text{ nm}$$

$$\lambda = 500 \text{ nm} \Rightarrow l_c = 2.5 \text{ mm}$$

② White light:

$$\Delta\lambda = 150 \text{ nm} \Rightarrow l_c = 2 \times 10^3 \text{ mm}$$

we are considering

$$\lambda = 500 \text{ nm}$$

$\times 4 \lambda \rightarrow$ this is the number of fringes

the maximum spectral sensitivity of the eye, which spans $4000 \text{ \AA} \leq \lambda \leq 7000 \text{ \AA}$

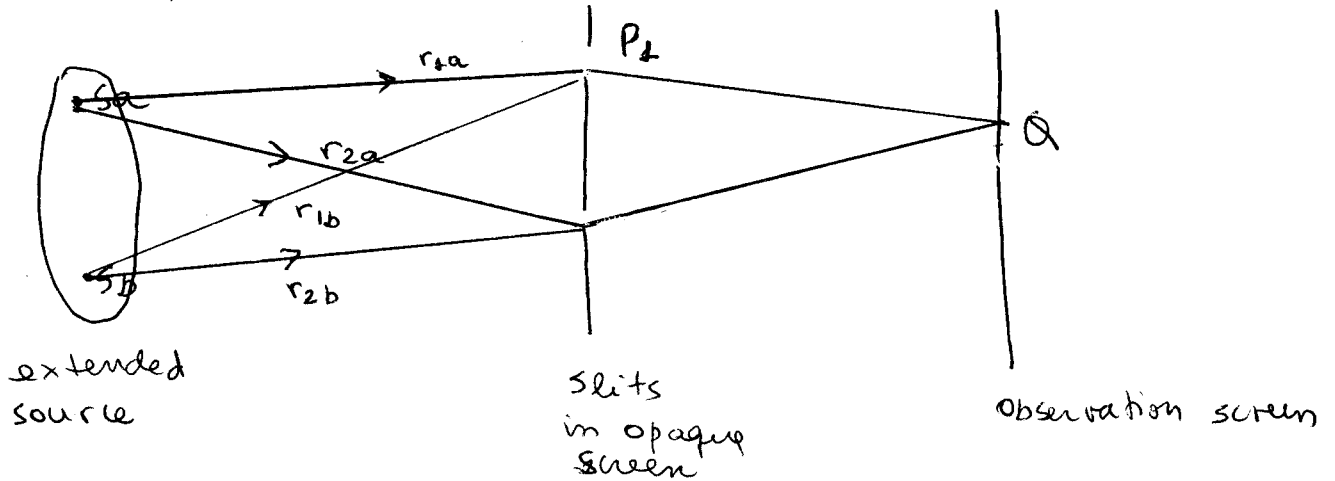
that can be seen, e.g., in a Michelson interferometer

3. (b) - Spatial coherence

• Coherence between two fields at different points in space

• Importance: extended sources

Let us consider the coherence of 2 points P_1, P_2 in the radiation field of an extended source



S_a, S_b are mutually incoherent and identical

The electric field at Q is $\vec{E} = \vec{E}_1 + \vec{E}_2$

$$E_1 = E_{1a} + E_{1b} \quad (\text{field at } P_1)$$

$$E_2 = E_{2a} + E_{2b} \quad (\text{field at } P_2)$$

Degree of partial coherence for the 2 receiving points

P_1, P_2 :

$$\gamma_{12}(z) = \frac{\langle E_1(t) E_2^*(t+z) \rangle}{\sqrt{I_1 I_2}} = \frac{\langle [E_{1a}(t) + E_{1b}(t)] [E_{2a}^*(t+z) + E_{2b}^*(t+z)] \rangle}{\sqrt{I_1 I_2}}$$

$$= \frac{\langle E_{1a}(t) E_{2a}^*(t+z) \rangle}{\sqrt{I_1 I_2}} + \frac{\langle E_{1b}(t) E_{2b}^*(t+z) \rangle}{\sqrt{I_2 I_1}} +$$

$$+ \frac{\langle E_{1a}(t) E_{2b}^*(t+z) \rangle}{\sqrt{I_1 I_2}} + \frac{\langle E_{1b}(t) E_{2a}^*(t+z) \rangle}{\sqrt{I_1 I_2}}$$

$= 0 \rightarrow S_a, S_b$ mutually incoherent
 $\leftarrow 0$
 $''$

We assume now that each field $E_{1a}, E_{1b}, E_{2a}, E_{2b}$ is of the form

$$E_{1a}(t) = E_{1a} e^{-i\omega t + i\phi(t)} \quad E_{1a} = E_{1b} = E_1$$

$$E_{2a}(t) = E_{2a} e^{-i\omega t + i\phi(t)} \quad E_{2a} = E_{2b} = E_2$$

$\phi \equiv$ phase changing randomly and stepwise at a time interval z_0

$$\frac{\langle E_{1a}(t) E_{2a}^*(t+z_a) \rangle}{\sqrt{I_1 I_2}} = \frac{E_1 E_2}{\sqrt{I_1 I_2}} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{-i\omega t + i\omega(t+z_a)} \times e^{i\phi(t) - i\phi(t+z_a)} dt$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{i\omega z_a} e^{i\phi(t) - i\phi(t+z_a)} dz$$

$$e^{i\omega z_a} \left[1 - \frac{z_a}{z_0} \right] = \frac{\gamma(z_a)}{2}$$

Similarly, $\frac{\langle E_{1b}(t) E_{2b}^*(t+z_b) \rangle}{\sqrt{I_1 I_2}} = \frac{\gamma(z_b)}{2}$

$$\gamma_{12}(z) = \frac{1}{2} [\gamma(z_a) + \gamma(z_b)]$$

time due to the path difference

$$z_a = \frac{r_{2a} - r_{1a}}{c} + z$$

$$z_b = \frac{r_{2b} - r_{1b}}{c} + z$$

residual time

$$|Y_{12}|^2 = \frac{1}{4} \left[e^{i\omega z_a} \left(1 - \frac{z_a}{z_0}\right) + e^{i\omega z_b} \left(1 - \frac{z_b}{z_0}\right) \right] \left[e^{-i\omega z_a} \left(1 - \frac{z_a}{z_0}\right) + e^{-i\omega z_b} \left(1 - \frac{z_b}{z_0}\right) \right]$$

$$\left[e^{i\omega z_b} \left(1 - \frac{z_b}{z_0}\right) \right] = \frac{1}{4} \left[2 \cos[\omega(z_a - z_b)] \left(1 - \frac{z_a}{z_0}\right) \left(1 - \frac{z_b}{z_0}\right) + \left(1 - \frac{z_a}{z_0}\right)^2 + \left(1 - \frac{z_b}{z_0}\right)^2 \right]$$

Summing and subtracting $2 \left(1 - \frac{z_b}{z_0}\right) \left(1 - \frac{z_a}{z_0}\right)$ we have

$$|Y_{12}|^2 = \left[\left[2 + 2 \cos[\omega(z_a - z_b)] \right] \left(1 - \frac{z_a}{z_0}\right) \left(1 - \frac{z_b}{z_0}\right) - 2 \left(1 - \frac{z_a}{z_0}\right) \left(1 - \frac{z_b}{z_0}\right) + \left(1 - \frac{z_a}{z_0}\right)^2 + \left(1 - \frac{z_b}{z_0}\right)^2 \right] \frac{1}{4}$$

$$(*) = \left[1 - \frac{z_a}{z_0} - 1 + \frac{z_b}{z_0} \right]^2 = \left[\frac{z_b - z_a}{z_0} \right]^2 \Rightarrow \text{can be neglected if } z_b - z_a \ll z_0 \ll z_a$$

$$\Rightarrow |Y_{12}|^2 = \frac{1}{2} \left[1 + \cos[\omega(z_a - z_b)] \right] \left(1 - \frac{z_a}{z_0}\right) \left(1 - \frac{z_b}{z_0}\right)$$

Consequently, the mutual coherence between the fields at P_1 and P_2 depend on

- ① The self coherence time z_0 at the radiation in the source
- ② The time difference $z_a - z_b$

