Engingeering Mathemathics III Summary Sheet 4

Eigenvalues and Eigenvectors

1 Trivial vs. Non-trivial

1. The system $AX = 0$ always possesses the **Trivial Solution** $X = 0$. Note: Here 0 is the column vector $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$
2. If $|A| \neq 0$ the system $AX = 0$ possesses **only** the trivial solution, since A^{-1} exists and the unique solution is $X = A^{-1}0 = 0$.
3. If $|A| = 0$ then $AX = 0$ also possesses a **Non-trivial Solution** in which some of the elements are non-zero. In these cases, there will be a family of solutions.

2 Eigenvalues

1. A matrix A has a set of eigenvalues λ defined by $AX = \lambda X$. Note: λ is a number.
2. Noting $\lambda X = \lambda IX$, the eigenvalue problem can be re-written $(A - \lambda I)X = 0$.
3. Non-trivial solutions of this equation require $|A - \lambda I| = 0$ which is an n^{th} order polynomial in λ known as the **Characteristic Equation**.
4. Thus there are n eigenvalues λ_i, $i = 1, \ldots, n$ corresponding to the n roots of the characteristic equation.

3 Eigenvectors

1. For each specific λ_i we can solve $(A - \lambda_i I)X_i = 0$ to find the non-trivial **eigenvector** X_i using any standard technique for simultaneous linear equations.
2. If there is a repeated root to the characteristic equation, there will be extra free parameters in the corresponding non-trivial X_i so that a full set of n eigenvectors can be found.

4 Diagonlisation

Modal Matrix is the matrix M the columns of which are the eigenvectors of A, thus

$$M = \left(\begin{pmatrix} X_1 \end{pmatrix} \begin{pmatrix} X_2 \end{pmatrix} \cdots \begin{pmatrix} X_n \end{pmatrix} \right)$$

Spectral Matrix is the matrix D which is diagonal, with the eigenvalues along the diagonal, i.e.,

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Diagonalisation of A is accomplished by $M^{-1}AM = D$ which can be seen by multiplying this equation on the left by M.

5 Special Properties

1. The eigenvalues of a real symmetric matrix are real.
2. The eigenvectors associated with the distinct eigenvalues of a real symmetric matrix are orthogonal (i.e., perpendicular).

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