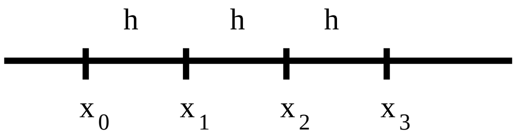
Engineering Mathemathics III Summary Sheet 9
Approximate Methods for ODEs

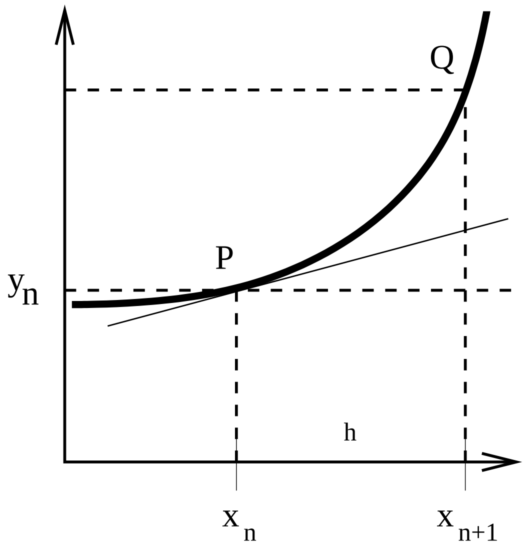
1 Numerical Solutions of First Order ODEs

1. We wish to solve the 1st order ODE
$$y' \equiv \frac{dy}{dx} = f(x,y)$$
given $y = y_0$ at $x = x_0$.

2. Divide the x-axis into sections of length h. Label these points x_0, x_1, $x_2 \ldots$:



3. Develop numerical schemes to estimate $y_{n+1} \equiv y(x_{n+1})$ from x_n,y_n and $f(x,y)$.



4. The simplest is *Euler's method*, which uses the slope at x_n to approximate the curve to a local straight line, as illustrated above and given by:
$$y_{n+1} = y_n + hf(x_n,y_n) \equiv y_n + hy'_n$$
The error reduces *per step* as h is reduced by $O(h^2)$.

5. *Modified Euler* is similar but uses the average of the slope of the function between x_n and x_{n+1} to improve the error behaviour to $O(h^3)$. The slope at x_{n+1} is estimated by using the Euler method:
Slope at P$= y'_n = f(x_n,y_n)$
Slope at Q$\approx g_n = f(x_n+h, y_n+y'_n)$
so
$$y_{n+1} = y_n + \frac{1}{2}h(y'_n + g_n)$$

6. *Runge-Kutta* goes further in refining the estimate to give a formulation with error per step $O(h^5)$, and results in
$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
where
$$\begin{aligned} k_1 &= hf(x_n,y_n) \\ k_2 &= hf(x_n+\frac{1}{2}h, y_n+\frac{1}{2}k_1) \\ k_3 &= hf(x_n+\frac{1}{2}h, y_n+\frac{1}{2}k_2) \\ k_4 &= hf(x_n+h, y_n+k_3) \end{aligned}$$

2 Power Series Approximate Solutions

2.1 Picard's Method for 1st order ODEs

To find an approximate power series solution to
$$\frac{dy}{dx} = f(x,y)$$
given $y = y_0$ at $x = x_0$

1. Integrate formally the ODE to give
$$y(x) = y_0 + \int_{x_0}^{x} f[\chi, y(\chi)]\, d\chi$$

2. Develop a scheme of successive approximate functions $y_n(x)$ using this integral equation, namely
$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f[\chi, y_n(\chi)]\, d\chi$$

2.2 Picard's Method for 2nd order ODEs

Equations of the form:
$$\frac{d^2y}{dx^2} = g(x, y, \frac{dy}{dx})$$
given $y = y_0$ and $\frac{dy}{dx} = y'_0$ at $x = x_0$. Re-write the 2nd order ODE as two 1st order ones by letting $z \equiv \frac{dy}{dx}$. This gives
$$\begin{aligned} \frac{dz}{dx} &= g[x, y(x), z(x)] \\ \frac{dy}{dx} &= z(x) \end{aligned}$$

Then solve the above two equations using Picard's method, i.e.,
$$\begin{aligned} z_{n+1}(x) &= y'_0 + \int_{x_0}^{x} g[\chi, y_n(\chi), z_n(\chi)]\, d\chi \\ y_{n+1}(x) &= y_0 + \int_{x_0}^{x} z_n(\chi) d\chi \end{aligned}$$

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