Engingeering Mathemathics III Summary Sheet 3

Simultaneous Linear Equations: Gaussian Elimination and LU Decomposition

1 Notation and Definitions

Row-echelon Form A matrix is in row-echelon form if:

1. Any rows with all zeroes are at the bottom
2. The 1st non-zero element in each row is 1
3. The 1st non-zero element in each row is to the right of that in the preceding row

The following is in row-echelon form:

$$\begin{pmatrix} 1 & 3 & -2 & 4 & 7 \\ 0 & 1 & 0 & 6 & -3 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Back Substitution is the process of taking an augmented matrix which is in row-echelon form and working from the bottom up, substituting the values obtained by solving the last equation to solve the next-to-last, etc.

2 Gaussian Elimination

2.1 Elementary Row Operations

A set of simultaneous equations, and therefore the augmented matrix which contains all the necessary information to find a solution, is unchanged by the following operations:

1. Interchanging any 2 rows
2. Multiplying or dividing any row by a non-zero constant
3. Adding or subtracting a multiple of a row to another row

2.2 Steps in Gaussian Elimination for Solving Simultaneous Linear Equations

1. Form the augmented matrix
2. Use elementary row operations to reduce it to row-echelon form
3. Use back-substitution to determine the solution

3 *LU* Decomposition

1. Re-write the system as $AX = LUX = B$
2. L is lower triangular. That is, it has all zeroes above the diagonal, so it is of the form

$$L = \begin{pmatrix} a & 0 & 0 & \dots \\ b & c & 0 & \dots \\ d & e & f & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

3. U is upper triangular (all zeroes below the diagonal) and can be taken to have 1's along the diagonal, e.g.:

$$\begin{pmatrix} 1 & \alpha & \beta & \dots \\ 0 & 1 & \gamma & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

4. Multiplying out LU and equating to A reveals that the unknowns $a, b, c, \dots \alpha, \beta, \dots$ can be found systematically starting with the $(1,1)$ element of A.
5. Now let $UX = Y$ so the original system is $LY = B$. Since L is triangular, this can be solved by forward-substitution starting at the top row, which has only one non-zero element, and then proceding downward.
6. Once Y is known, the system $UX = Y$ can be solved easily by back-substitution.
7. This method has many steps, but each one is straightforward and systematic. *LU* decomposition is well-suited to numerical coding.

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