Engingeering Mathemathics III Summary Sheet 2

Simultaneous Linear Equations: Properties of Solutions

1 Notation

Consider the set of n equations in n unknowns $x, y, z, \ldots$

$$\begin{aligned} a_1x + b_1y + c_1z + \ldots &= k_1 \\ a_2x + b_2y + c_2z + \ldots &= k_2 \\ \vdots \quad &= \vdots \\ a_nx + b_ny + c_nz + \ldots &= k_n \end{aligned}$$

In matrix notation, we can write this system as $AX = B$ where

$$A = \begin{pmatrix} a_1 & b_1 & c_1 & \ldots \\ a_2 & b_2 & c_2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \\ \vdots \end{pmatrix}, B = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ \vdots \end{pmatrix}$$

2 Existence of Solutions

A pair of simultaneous linear equations in two unknowns correspond to the intersection of two straight lines. From this geometrical perspective it is clear that three possibilities exist:

1. There is a unique solution (corresponding to the intersection point).
2. There is no solution (if the lines are parallel and distinct).
3. There is an infinite number of solutions (a "family") (if the 2 lines lie on top of one another).

These three possibilities (unique solutions, no solution, infinite families) exists for systems of any size.

Rank of a matrix is size of largest sub-matrix which has non-zero determinant.

Linear Independence means no linear combination yields zero

Linear Dependence means some linear combination yields zero

Augmented Matrix is an $n \times (n+1)$ matrix formed from A by appending an additional column filled with B, i.e.,

$$\begin{pmatrix} a_1 & b_1 & c_1 & \ldots & k_1 \\ a_2 & b_2 & c_2 & \ldots & k_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

2.1 Unique Solutions

1. $|A| \neq 0$ necessary and sufficient ($\Rightarrow$ Rank of $A =$ Rank of the Augmented Matrix $=$ number of variables)
2. Cramer's Rule: $x = \dfrac{\begin{vmatrix} k_1 & b_1 & c_1 & \ldots \\ k_2 & b_2 & c_2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}}{|A|}$
3. Similarly for y, etc. with the corresponding column of A replaced by the column vector B
4. $X = A^{-1}B$

2.2 Infinite Number of Solutions

1. $|A| = 0$ AND Equations Consistent
2. LH Sides Linearly Dependent; RH Sides have same dependency
3. Rank of A is the same as the Rank of the Augmented Matrix and is less than the number of variables
4. Any ONE of the above implies all the others
5. 2×2: $\alpha \times$ 1st equation (LH & RH sides) $=$ 2nd equation
6. $n \times n$: $\sum_j \alpha_j (j_{th}\text{equation})$ gives zero on both LH and RH sides for some set of α_j's.
7. Number of free parameters $=$ number of variables (rows) minus Rank of A.

2.3 No Solutions

1. $|A| = 0$ AND Equations Inconsistent
2. LH Sides Linearly Dependent; RH Sides have different dependency
3. Rank of A is less than the Rank of the Augmented Matrix
4. Any ONE of the above implies all the others
5. 2×2: $\alpha \times$ LH Side of 1st equation $=$ LH Side of 2nd equation but $\alpha k_1 \neq k_2$
6. $n \times n$: A set of α_j's exist such that $\sum_j \alpha_j (j_{th}\text{equation})$ gives zero on LH side but NOT on RH side.

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