SUMMARY SHEET: PARTIAL DIFFERENTIATION
Engineering Maths II (MAE111), 2003

Fundamental definition: For function $f = f(x, y)$ of two variables x and y, the partial derivative wrt x is:

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

Note that this is effectively the rate of change of f in the x direction.

Evaluation: Use the same rules as ordinary differentiation. When differentiating wrt one variable (say x), then treat other variables (say y) as constant.

Example: For

$$z = f(x, y) = 4x^2 + 3xy + 5y^2$$

the first order partial derivatives are

$$\begin{aligned} \frac{\partial z}{\partial x} &= 8x + 3y \\ \frac{\partial z}{\partial y} &= 3x + 10y \end{aligned}$$

Higher Order Derivatives: For example, second order differential coefficients:

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}.$$

Note: for most functions (actually continuous functions), the order of partial differentiation is unimportant:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Small Increments: The approximate change in a function $z(x, y)$ due to small changes in x and y (denoted by δx, δy) is given by

$$\delta z \approx \frac{\partial z}{\partial x}\, \delta x + \frac{\partial z}{\partial y}\, \delta y.$$

This can be used for error estimation for a calculated quantity which depends on measurements with some fractional errors.

Total Derivative: For $f(x, y)$ and if $x = x(t)$ and $y = y(t)$, where t is another variable, then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

Note mixture of partial and ordinary derivatives.

Change of Variables: Consider function $z = z(x, y)$, where the variables x and y are themselves functions of another pair of variables u and v, so that

$$x = x(u, v) \quad \text{and} \quad y = y(u, v).$$

If z is considered as a function of u and v then

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \end{aligned}$$

Stationary Points: For $f(x, y)$ a stationary point is defined as where the rate of change is zero, so that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$$

A stationary point may be minimum, maximum, or some other kind such as a saddle point.

For a MAXIMUM:

$$\frac{\partial^2 z}{\partial x^2} < 0 \quad \text{AND} \quad \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$$

For a MINIMUM:

$$\frac{\partial^2 z}{\partial x^2} > 0 \quad \text{AND} \quad \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$$

For a SADDLEPOINT:

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 < 0$$

(i.e., the function decreases in some directions, but increases in others).

Constrained Maxima and Minima: In order to find a maximum or minimum of a function of several variables, which **also** satisfies a further constraint: *use the constraint (or constraints) to eliminate one (or more) of the variables. And then look for maxima/minima.*

Example: A cylindrical tin is to be made out of 600π cm^2 of sheet metal. What choice of radius r and height h will give the tin a maximum volume? Volume: $V = \pi r^2 h$, surface area (including ends)

$$S = 2\pi r h + 2\pi r^2 \quad \Rightarrow h = \frac{S}{2\pi r} - r$$

Substituting for h into equation for V:

$$V = \pi r^2 \left(\frac{S}{2\pi r} - r\right) = \frac{Sr}{2} - \pi r^3$$

Then find value of r giving maximum V (remembering that S is fixed. And so on ...

Curve Fitting by Least Squares: For a set of n data points (x_i, y_i), with $i = 1, \ldots n$, choose a suitable curve to "fit" the data, eg a straight line $y = a + bx$ (but it could be some other function). For each data point compute the y distance from the curve, and then square and then add up for all the points:

$$S = \sum_{i=1}^{n} (y_i - a - bx_i)^2 = S(a, b)$$

The best ("least squares") fit is when the function $S(a, b)$ is a minimum. Therefore the best fit is given by the values of a and b for which S is stationary:

$$\frac{\partial S}{\partial a} = 0, \quad \frac{\partial S}{\partial b} = 0.$$

Text Book References

	Mathematics for Engineers Croft & Davison	Engineering Mathematics Stroud (4th & 5th Editions)
Partial Differentiation	–	Programme 10, 11