Engingeering Mathemathics III Summary Sheet 7

Ordinary Differential Equations: 3

1 Euler Equations

Equations of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 x \frac{dy}{dx} + a_0 y = f(x)$$

are called Euler Equations. These equations do NOT have constant coefficients.

1. Euler Equations can be transformed into such by the substitution $t = \ln x$, $x = e^t$
2. $\frac{d}{dx} = \frac{d}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{d}{dt}$
3. Thus $\frac{d}{dt} = x\frac{d}{dx}$
4. Then $\frac{d^2}{dt^2} = \frac{d}{dt}\frac{d}{dt} = x\frac{d}{dx}\left(x\frac{d}{dx}\right) = x\frac{d}{dx} + x^2\frac{d^2}{dx^2} = \frac{d}{dt} + x^2\frac{d^2}{dx^2}$
5. Thus $x^2\frac{d^2}{dx^2} = \frac{d^2}{dt^2} - \frac{d}{dt}$
6. By repeated application of this derivation all terms $x^m \frac{d^m y}{dx^m}$ can be transformed into a sum of terms involving derivatives with respect to t all of which have constant coefficients.
7. Re-write $f(x) = f(e^t)$ as a function of t.
8. Solve the Linear ODE in t which has constant coefficients using previous methods.
9. Re-write the solution in terms of x and apply boundary conditions if any.

2 Simultaneous Linear ODEs

These are of the form

$$\begin{aligned} F(\mathcal{D})y + f(\mathcal{D})z &= H(x) \quad (1) \\ G(\mathcal{D})y + g(\mathcal{D})z &= J(x) \quad (2) \\ & \quad (3) \end{aligned}$$

where the notation $\mathcal{D} \equiv \frac{d}{dx}$ is just the derivative operator, and F, f, G, and g are polynomial expressions involving $\mathcal{D}$, e.g., $F(\mathcal{D}) = a\mathcal{D}^2 + b\mathcal{D} + c$.

These systems can be solved by:

1. Taking $G(\mathcal{D})$ of Equation 1 and $F(\mathcal{D})$ of Equation 2.
2. Taking the difference of the result, which therefore eliminates the y terms to leave an ODE in z which is linear with constant coefficients
3. Solving this ODE for $z(x)$
4. Using this solution to then find $y(x)$ from the original equations.
5. In doing this, it is best to try to eliminate the $\mathcal{D}^n y$ terms between Equations 1 and 2 to leave an equation which can be solved algebraically for $y(x)$ in terms of the known function $z(x)$ and its derivatives. If this is not possible, there is a risk that differentiating somewhere will introduce an extra arbitrary constant in the final solution. This problem can be resolved by substituing the final results for y and z back into the original two equations to verify that they are solutions, finding conditions on any arbitrary constants which are necessary to make them so.

3 Systems of 1st Order ODEs

These are of the form

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots &= \vdots \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{aligned}$$

which can be cast into the matrix form

$$\dot{x} = Ax$$

where $\dot{x} \equiv \frac{dx}{dt}$. To solve these

1. Find the eigenvalues and eigenvectors, and hence the modal matrix M corresponding to A.
2. Let $x = My$. Then $\dot{x} = M\dot{y} = Ax = AMy$. Multiplying by M^{-1} gives $\dot{y} = M^{-1}AMy \equiv Dy$ where D is the spectral matrix, which is diagonal with the eigenvalues of A along the diagonal.
3. Thus the system separates into n independent equations $\dot{y}_i = \lambda_i y_i$ with solutions $y_i = C_i e^{\lambda_i t}$
4. Once y is known, $x = My$ gives the final solution

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