Engingeering Mathemathics III Summary Sheet 6

Ordinary Differential Equations: 2

1 Second Order Linear ODEs with Constant Coefficients

Equations of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

Because the equation is linear, it's solution can be put together from pieces, e.g., $y=y_{CF}+y_{PI}$ where y_{CF} is the **Complementary Function**, which satisfies the ODE with zero on the right hand side and y_{PI} is a **Particular Integral** which is ANY function y which when substituted into the LHS of the ODE yields $f(x)$.

1.1 Complementary Function

1. y_{CF} is the general solution to

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

which is known as the **Homogeneous Equation**.

2. Guess $y_{CF}=e^{kx}$ leads to the **Auxiliary Equation**

$$ak^2+bk+c=0$$

3. Roots of the Auxiliary Equation are k_1 and k_2.

4. Cases:

k_1, k_2	y_{CF}
real, distinct	$Ae^{k_1x}+Be^{k_2x}$
complex: $\alpha\pm\beta j$	$e^{\alpha x}(C\cos\beta x+D\sin\beta x)$
equal: $k_1=k_2$	$(A+Bx)e^{k_1x}$

1.2 Particular Integral

y_{PI} is ANY solution to

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

It will have NO arbitrary constants. y_{PI} is found by guessing the form of y_{PI} from the form of $f(x)$ and then finding the unknown parameters in the guess:

$f(x)$	trial y_{PI}
Constant	Constant
$ax^r+bx^{r-1}+\ldots+dx+e$	$\alpha x^r+\beta x^{r-1}+\ldots+\delta x+\varepsilon$
$a\cos kx$	$\alpha\cos kx+\beta\sin kx$
$a\sin kx$	$\alpha\cos kx+\beta\sin kx$
ae^{kx}	αe^{kx}
$a\cosh kx$	$\alpha\cosh kx+\beta\sinh kx$
$a\sinh kx$	$\alpha\cosh kx+\beta\sinh kx$
ae^{k_1x} (part of y_{CF})	αxe^{k_1x}

1.3 Boundary Conditions

The general solution to a 2nd order linear ODE with constant coefficients, $y=y_{CF}+y_{PI}$ has two arbitrary constants in y_{CF}. These can be evaluated if (two) boundary conditions are given. Typically, such boundary conditions are the values y_1 and y_2 of y at two points x_1 and x_2 or the values y_0 and y'_0 of y and $\frac{dy}{dx}$ at one point x_0. Note that the boundary conditions are applied to the general solution, e.g.,

$$y_0=y_{CF}(x_0)+y_{PI}(x_0)$$

2 Higher Order Linear ODEs with Constant Coefficients

Equations of the form

$$a_n\frac{d^ny}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1\frac{dy}{dx}+a_0y=f(x)$$

are solved in a similar fashion:

1. $y=y_{CF}+y_{PI}$

2. Auxiliary Equation becomes

$$a_nk^n+a_{n-1}k^{n-1}+\ldots+a_1k+a_0=0$$

with n roots. For a_i's all real, any complex roots for k always occur in complex conjugate pairs.

3. $y_{CF}=A_1e^{k_1x}+A_2e^{k_2x}+\ldots+A_ne^{k_nx}$ which may be re-written in terms of sines and cosines in the case of complex roots.

4. If a root, say k_1, is repeated r times, the component of y_{CF} corresponding to this root is $\left(A_1+A_2x+\ldots+A_rx^{r-1}\right)e^{k_1x}$

5. y_{PI} is again found by trial forms as in the table in the preceding section.

6. If $f(x)$ corresponds to a part of the complementary function for which the root, say k_1, is repeated r times ($r\geq 1$) the trial form is $y_{PI}=\alpha x^re^{k_1x}$.

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