Engingeering Mathemathics III Summary Sheet 5
Ordinary Differential Equations: 1

1 Definitions and Concepts

Ordinary Differential Equation ("ODE") is an equation involving the derivatives with respect to a single variable.

Independent Variable does not require knowledge of the value of other variables.

Dependent Variable depends on the value of the independent variable. E.g., $y(x) = x^2 + 7$ or $\frac{dy}{dx} = 2x$ have x as the independent variable, and y as the dependent variable. $\frac{d^2x}{dt^2} + 3tx = 0$ has x as the dependent variable and t as the independent variable.

Order of a differential equation is the order of the highest derivative appearing in the equation. $\left(\frac{d^2y}{dx^2}\right)^7 = 3x$ is of order 2.

Linear ODEs have terms which contain only the dependent variable (y say) or its derivative not raised to any power nor in combination with each other nor within a nonlinear function. $\frac{d^2y}{dx^2} + 3e^x y + 7 = 0$ is linear; $\sqrt{y\frac{dy}{dx}} + 3y^2 = 2x$ is not.

Constant Coefficient Linear equations are linear equations in which each term involving the dependent variable or its derivatives is multiplied only by a constant, not some function of the dependent variable. E.g. $2\frac{d^2y}{dx^2} + \frac{dy}{dx} + 7y = 3x$.

General Solutions to a ODEs have the same number of arbitrary constants as the **order** of the ODE.

Particular Solutions have such constants chosen so as to meet various conditions on the dependent variable and/or its derivatives. Conditions often come in the form of **Initial Conditions** at the minimum value of the independent variable. Conditions imposed at either extrema of the independent variable are called **Boundary Conditions**.

2 Methods of Solution: 1st Order ODEs

For simplicity, in everything which follows, we shall use x as the independent variable, and y as the dependent one.

2.1 Basic Methods

1. **Direct Integration** is sometimes possible. Note that $\int \frac{dy}{dx}\,dx = \int dy = y$ and remember to include a constant of integration.
2. **Separation of Variables** can always be done for equations which can be cast in the form $\frac{dy}{dx} = f(x)g(y)$ to give $\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$
3. **Homogeneous Equations** can be cast into the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ and include equations of the form $\frac{dy}{dx} = \frac{(a_1x+b_1y)^r}{(a_2x+b_2y)^r}$. These can be solved by the substitution $y = vx$.
4. **Reducible to Homogeneous** are equations of the form $\frac{dy}{dx} = \frac{(a_1x+b_1y+c_1)^r}{(a_2x+b_2y+c_2)^r}$ and can be solved by changing variables to $x = X+h$, $y = Y+k$ and finding h and k which makes the resulting equation homogeneous in X and Y.
5. If the previous method fails, because there is no solution for h and k, then the equation can be written $\frac{dy}{dx} = \frac{(a_1x+b_1y+c_1)^r}{[L(a_1x+b_1y)+c_2]^r}$ which can be solved by the substitution $a_1x+b_1y = u$

2.2 Exact Equations

1. A 1st order ODE can always be written $R(x,y)\frac{dy}{dx} + S(x,y) = 0$.
2. This equation is said to be **exact** if one can find a function $u[y(x),x]$ such that the equation can be re-written $\frac{du}{dx} = 0 = \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial x}$.
3. The solution is then $u[y(x),x] = \text{constant}$.
4. Testing for exactness: $\frac{\partial R}{\partial x} = \frac{\partial S}{\partial y}$. If so, then $\frac{\partial u}{\partial y} = R$ and $\frac{\partial u}{\partial x} = S$ which can be integrated to find u.

2.2.1 Integrating Factor for Linear 1st order ODEs

1. Cast the equation into the standard form $\frac{dy}{dx} + P(x)y = Q(x)$.
2. Multiply the equation by the **Integrating Factor** $\mu = e^{\int P\,dx}$.
3. Equation is now $\frac{d}{dx}(\mu y) = \mu Q$ which can be integrated.

2.2.2 Equations of Bernoulli Type

1. Equations of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$
2. Substitute $z = y^{1-n}$ to yield a linear 1st order ODE for z which can be solved by finding an Integrating Factor.

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