Engingeering Mathemathics III Summary Sheet I

Matrices and Determinants

1 Matrices

1. Definition: A rectangular array, e.g.,

$$A = \begin{pmatrix} 1 & 7 \\ 2 & -3 \\ 4 & 13 \end{pmatrix}$$

is a 3×2 matrix ($n \times m \equiv rows \times cols$)

2. Notation: $A \equiv \mathbf{A} \equiv \underline{\underline{A}} \equiv (a_{ij})$.

3. Equality: $A = B$ requires $a_{ij} = b_{ij}$ for all i, j. Thus A and B must be the same size.

4. Special Matrices:

Square $n = m$

Identity $\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$

Transpose A^T Interchanges rows and cols

5. Arithmetic: $C = A \pm B$
   - A, B must both same $n \times m$
   - Element by element, i.e., $c_{ij} = a_{ij} \pm b_{ij}$
   - $A + Z = A$ in general only for $Z =$ the zero matrix, with all elements zero.

6. $(A+B)^T = A^T + B^T$

7. $kA = (ka_{ij})$

8. Matrix Multiplication $\underset{n \times m}{A} \; \underset{p \times q}{B} = \underset{n \times q}{D}$
   REQUIRES $m \equiv p$

9. For $D = AB$, $d_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$

10. $(AB)^T = B^T A^T$

11. E.g. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} =$
$\begin{pmatrix} 1\times5+2\times8 & 1\times6+2\times9 & 1\times7+4\times10 \\ 3\times5+4\times8 & 3\times6+4\times9 & 3\times7+4\times10 \end{pmatrix}$

12. In general $AB \neq BA$, even if both are possible

13. $AI = A = IA$ where I is appropriate size

2 Determinants

2.1 Definitions

1. Determinant is a number associated with a SQUARE matrix

2. Notation $\det A \equiv |A| \equiv \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{vmatrix}$

3. **Minor** m_{ij} related to a_{ij} is determinant of $(n-1) \times (n-1)$ matrix found by removing row i and col j of A

4. **Cofactor** $c_{ij} = (-1)^{i+j} m_{ij}$

5. Sign factor $(-1)^{i+j}$ from $\begin{pmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

6. $\det(A) = \sum_{\text{any 1 row or col}} a_{ij} c_{ij}$
   i.e., sum over any row or col of each element times its cofactor.

7. A **Singular Matrix** is one whose determinant is zero.

2.2 Properties of Determinants Helpful in Evaluation

1. Form B from A by multiplying any row (or col) of A by k, then $|B| = k|A|$

2. $|kA| = k^n |A|$

3. $\det(A^T) = \det(A)$

4. Form B from A by interchanging any 2 rows (or cols) of A. Then $|B| = -|A|$

5. Form B from A by adding a multiple of any row (col) to another row (col). Then $|B| = |A|$

6. For A, B both $n \times n$, $|AB| = |A||B|$

7. If any 2 rows (or cols) of A are equal, $|A| = 0$

3 Matrix Inversion

1. If A is $n \times n$ and $|A| \neq 0$ then A^{-1} exists such that $A^{-1}A = I = AA^{-1}$. If $|A| = 0$ then A^{-1} does not exist.

2. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \dfrac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

3. For A an $n \times n$ matrix, its **adjoint**, $\text{adj}(A)$, is the matrix of cofactors of A^T

4. Then $A^{-1} = \dfrac{\text{adj}(A)}{|A|}$ provided $|A| \neq 0$

5. If some linear combination of rows (or cols) of A is zero, $|A| = 0$

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