Dynamics of Physical Systems (MTH5106)

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A brief history

• **Aim of Science:** describe & predict evolution of multiplicity of phenomena, including universe itself!

• **Physical Systems:** important subset of systems in the universe, ranging from **falling apples, raindrops, planets to the Universe itself!**

• **Dynamics:** scientific study of changes/motion in physical (and other!) systems.

One of the first exact sciences to develop.

Deals with motion of macroscopic bodies.

Modern developments from 17 Century:

Galileo (1564-1642): 1st to show with help of experiments that under const force a body moves with cons accel and not a const vel as Aristotle had believed!

Newton (1642-1727): completed laws of Mechanics; introduced concept of mass; gave law of gravitation; helped develop calculus!

And then came:

d'Alambert,
Euler,
Lagrange,
Poincare,
Einstein,
Kolmogorov, Smale ...

CRUCIAL POINT: SIMPLE MATHEMATICAL LAWS CAN TELL US SO MUCH ABOUT PHENOMENA ON SUCH VAST RANGE OF SCALES

1 Basic concepts in dynamics

Physical phenomena in the Universe occur on **enormous ranges of scales in space and time.** Classical dynamics deals with motion and change on common sense (intermediate) scales.

An important feature of the Universe is that **laws of dynamics obtained on earth seem applicable to phenomena on a vast range of scales**.

1.1 Frames/coordinates

To study motion one requires:

• **Frame of reference (FR):** i.e. a coordinate system (an origin in space + 3 orthogonal axes) plus a clock, in order to measure distances with time.

• **Cartesian and polar coordinates:** To specify a point in 3D we need 3 coordinates: (x, y, z) in **Cartesian coordinates** and (r, θ, ϕ) in **spherical polar coordinates.**

We often deal with motion in 1 or 2D. **In 2D we need 2 coords:** (x, y) **in Cartesian coordinates and** (r, θ) **in polar coordinates.**

In 2D the Polar and Cartesian coords are related by

$$x = r\cos\theta$$
$$y = r\sin\theta$$

The inverse transformations are given by

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

2 Vectors in brief

A **scalar** is a quantity with magnitude only $(p, q, s...)$

A **vector** is a quantity which has both a magnitude and a direction (represented by an underline: $\underline{a}, \underline{b}$, ... with magnitudes $|\underline{a}|$, $|\underline{b}|$, ...

Two vectors are equal if they have have same magnitude and direction.

A vector with magnitude but opposite direction is written as $-\underline{a}$

A **unit vector** along a given vector $\underline{a}$ (represented by a hat) is defined as

$$\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|}$$

2.1 Sum or resultant of 2 vectors

Sum of 2 vectors $\underline{a}$ and $\underline{b}$ is a vector $\underline{c} = \underline{a} + \underline{b}$.

2.2 Component representation

Any vector can be represented by its components.

In a Cartesian coordinate system xyz, origin at O, let $\underline{i}, \underline{j}, \underline{k}$ be unit vectors along the x, y and z axes.

Consider a vector $\underline{a}$ originating at O with coordinates of its end point $P = (a_1, a_2, a_3)$.

Then we can represent $\underline{a}$ as

$$\underline{a} = \underline{i}a_1 + \underline{j}a_2 + \underline{k}a_3$$

and

$$|\underline{a}| = \left({a_1}^2 + {a_2}^2 + {a_3}^2\right)^{1/2}$$

2.2.1 Addition of vectors in components

Let $\underline{a} = \underline{i}a_1 + \underline{j}a_2 + \underline{k}a_3$ and $\underline{b} = \underline{i}b_1 + \underline{j}b_2 + \underline{k}b_3$.

The sum of the two vectors is

$$\underline{a} + \underline{b} = \underline{i}(a_1 + b_1) + \underline{j}(a_2 + b_2) + \underline{k}(a_3 + b_3)$$

Example Let $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$ and $\underline{b} = 2\underline{i} + 3\underline{j} + 2\underline{k}$.

Then $\underline{a} + \underline{b} = 3\underline{i} + 5\underline{j} + \underline{k}$.

2.3 Laws of vector algebra

Consider vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ and scalars p, q. Then

$$\begin{aligned}
\underline{a} + \underline{b} &= \underline{b} + \underline{a} \\
p(q\underline{a}) &= pq\underline{a} \\
(p+q)\underline{a} &= p\underline{a} + q\underline{a} \\
\underline{a} + (\underline{b} + \underline{c}) &= (\underline{a} + \underline{b}) + \underline{c} \\
p(\underline{a} + \underline{b}) &= = p\underline{a} + q\underline{b}
\end{aligned}$$

Can see geometrically!
e.g. commutative rule of addition:

2.4 Vector multiplication

2.4.1 Scalar or dot product

for 2 vectors $\underline{a}$ and $\underline{b}$ defined as $\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos \theta$.

Geometrically

It follows that **if 2 vectors are perpendicular then** $\underline{a} \cdot \underline{b} = 0$ since $\theta = \pi/2$.

Also $\underline{a} \cdot \underline{a} = |\underline{a}|^2$ and $|\hat{\underline{a}}| \cdot |\hat{\underline{a}}| = 1$.

It also follows that [CHECK!]

$$\begin{aligned} \underline{i} \cdot \underline{i} &= \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1 \\ \underline{i} \cdot \underline{j} &= \underline{i} \cdot \underline{k} = \underline{j} \cdot \underline{k} = 0 \end{aligned}$$

Hence in component form

$$\begin{aligned}\underline{a}\cdot\underline{b} &= \left(\underline{i}a_1+\underline{j}a_2+\underline{k}a_3\right)\cdot\left(\underline{i}b_1+\underline{j}b_2+\underline{k}b_3\right)\\ &= a_1b_1+a_2b_2+a_3b_3\end{aligned}$$

Also for vectors $\underline{a}$, $\underline{b}$ and scalar p we have:

$$\begin{aligned}\underline{a}\cdot\underline{b} &= \underline{b}\cdot\underline{a}\\ (p\underline{a})\cdot\underline{b} &= p\underline{a}\times(\underline{b})\\ \underline{a}\cdot(\underline{b}+\underline{c}) &= \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}\end{aligned}$$

Example: Determine whether $\underline{a} = 2\underline{i}+\underline{j}-\underline{k}$ is perpendicular to $\underline{b}=\underline{i}+3\underline{j}+5\underline{k}$.

Solution: check that $\underline{a}\cdot\underline{b}$ =0.

Example: Find a vector that is perpendicular to the vector $\underline{a} = 2\underline{i} - 3\underline{j}$ in the xy plane. Is this vector unique?

Solution: Let this vector have the form
$\underline{b} = (b_1, b_2, 0)$.

Orthogonality implies $\underline{a} \cdot \underline{b} = 0$, which gives $2b_1 - 3b_3 = 0$ or $b_1 = 3/2b_2$. Clearly there are infinite such vectors!

2.4.2 Vector or cross product

for 2 vectors $\underline{a}$ and $\underline{b}$ defined as $\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin\theta \hat{n}$, where $\hat{n}$ is a unit vector such that $\underline{a}$, $\underline{b}$ and $\hat{n}$ form a right handed set.

Geometrically represents the area of parallelogram

It follows that if 2 vectors are parallel then $\underline{a} \times \underline{b} = 0$ since $\theta = 0$.

Also have (for vectors $\underline{a}$, $\underline{b}$ and scalar p)

$$\begin{aligned} \underline{a} \times \underline{b} &= -\underline{b} \times \underline{a} \\ (p\underline{a}) \times \underline{b} &= \underline{a} \times (p\underline{b}) \\ \underline{a} \times (\underline{b} + \underline{c}) &= \underline{a} \times \underline{b} + \underline{a} \times \underline{c} \end{aligned}$$

It also follows that [CHECK!]

$$\begin{aligned} \underline{i} \times \underline{i} &= \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = 0 \\ \underline{i} \times \underline{j} &= \underline{k}, \quad \underline{j} \times \underline{k} = \underline{i}, \quad \underline{k} \times \underline{i} = \underline{j} \end{aligned}$$

Hence in component form [CHECK!]

$$\begin{aligned} \underline{a} \times \underline{b} &= \\ \underline{i}\,(a_2 b_3 - a_3 b_2) &+ \underline{j}\,(a_3 b_1 - a_1 b_3) + \underline{k}\,(a_1 b_2 - a_2 b_1) \end{aligned}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: Find the cross product of the vectors $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = \underline{i} + 3\underline{j} + 5\underline{k}$.

Solution:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & 3 & 5 \end{vmatrix} = 8\underline{i} - 11\underline{j} + 5\underline{k}$$

2.5 Differentiation of vectors

Let $\underline{a}(t)$ be a vector function of variable t say.

Similar to diff of scalar functions we define

$$\frac{d\underline{a}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\underline{a}(t+\Delta t) - \underline{a}(t)}{\Delta t}$$

Expressed in terms of components

$$\begin{aligned} \underline{a}(t) &= \underline{i}a_1(t) + \underline{j}a_2(t) + \underline{k}a_3(t) \\ \underline{a}(t+\Delta t) &= \underline{i}a_1(t+\Delta t) + \underline{j}a_2(t+\Delta t) + \underline{k}a_3(t+\Delta t) \end{aligned}$$

Now since $\underline{i}, \underline{j}, \underline{k}$ are fixed vectors independent of time, differentiating component by component gives

$$\frac{d\underline{a}(t)}{dt} = \underline{i}\frac{d}{dt}(a_1(t)) + \underline{j}\frac{d}{dt}(a_2(t)) + \underline{k}\frac{d}{dt}(a_3(t))$$

Product rules of differentiation for vectors

Let $\underline{a}(t) = \underline{i}a_1 + \underline{j}a_2 + \underline{k}a_3$ and $\underline{b}(t) = \underline{i}b_1 + \underline{j}b_2 + \underline{k}b_3$

Then we have the following product rule of differentiation for vectors:

$$\begin{aligned}\frac{d}{dt}(\underline{a}\cdot\underline{b}) &= \underline{a}\cdot \mathrm{d}\underline{b}/\mathrm{d}t + \mathrm{d}\underline{a}/\mathrm{d}t\cdot\underline{b} \\ \frac{d}{dt}(\underline{a}\times\underline{b}) &= \underline{a}\times \mathrm{d}\underline{b}/\mathrm{d}t + \mathrm{d}\underline{a}/\mathrm{d}t\times\underline{b}\end{aligned}$$

which can be proved by writing $\underline{a}\cdot\underline{b}$ and $\underline{a}\times\underline{b}$ in terms of components. For example:

$$\begin{aligned}\frac{d}{dt}(\underline{a}\cdot\underline{b}) &= \frac{d}{dt}(a_1b_1 + a_2b_2 + a_3b_3) \\ &= \left(\frac{da_1}{dt}\right)b_1 + a_1\left(\frac{db_1}{dt}\right) + \left(\frac{da_2}{dt}\right)b_2 \\ &+ a_2\left(\frac{db_2}{dt}\right) + \left(\frac{da_3}{dt}\right)b_3 + a_3\left(\frac{db_3}{dt}\right) \\ &= \left(\frac{da_1}{dt}\right)b_1 + \left(\frac{da_2}{dt}\right)b_2 + \left(\frac{da_3}{dt}\right)b_3 \\ &+ a_1\left(\frac{db_1}{dt}\right) + a_2\left(\frac{db_2}{dt}\right) + a_3\left(\frac{db_3}{dt}\right) \\ &= \underline{a}\cdot \mathrm{d}\underline{b}/\mathrm{d}t + \mathrm{d}\underline{a}/\mathrm{d}t\cdot\underline{b}\end{aligned}$$

Similarly can prove

$$\frac{d}{dt}\left(\underline{a} \times \underline{b}\right) = \underline{a} \times \mathrm{d}\underline{b}/\mathrm{d}t + \mathrm{d}\underline{a}/\mathrm{d}t \times \underline{b}$$

by writing $\underline{a} \times \underline{b}$ in terms of components.

Example: A vector has constant magnitude but a direction that varies with time. Show that its derivative is always perpendicular to itself.

Solution: Let the vector be $\underline{c}$.

We know $\underline{c} \cdot \underline{c} = |\underline{c}|^2$ =constant.

$$\frac{d}{dt}[\underline{c} \cdot \underline{c}] = 0 = \underline{c} \cdot \frac{d\underline{c}}{dt} + \frac{d\underline{c}}{dt} \cdot \underline{c}$$

which gives

$$\underline{c} \cdot \frac{d\underline{c}}{dt} = 0$$

and hence the result.

2.6 Integration of vectors

For a vector function of time $\underline{a}(t) = \underline{i}a_1(t) + \underline{j}a_2(t) + \underline{k}a_3(t)$ we define the indefinite integral:

$$\int \underline{a}(t)dt = \underline{i}\int a_1(t)dt + \underline{j}\int a_2(t)dt + \underline{k}\int a_3(t)dt$$

Example Consider the vector

$$\underline{a}(t) = 2t\underline{i} + 3t^2\underline{j} + 4t^3\underline{k}$$

Calculate the derivative and the integral of this vector with respect to (wrt) t.

$$\frac{d\underline{a}(t)}{dt} = 2\underline{i} + 6t\underline{j} + 12t^2\underline{k}$$

$$\int \underline{a}(t)dt = t^2\underline{i} + t^3\underline{j} + t^4\underline{k} + \underline{C}$$

where $\underline{C}$ is a constant vector of integration.

2.7 Position vector

Let P be a particle moving relative to a Cartesian frame of reference F. Let its coordinates at time t be given by (x, y, z).

The vector connecting the origin to P is called the **position vector** of P relative to O and represented by $\underline{r}$:

$$\underline{r} = \underline{i}x + \underline{j}y + \underline{k}z$$

Now as the particle moves, its coordinates change with time and we can write

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

and we may represent $\underline{r}$ as a vector function of time, $\underline{r} = \underline{r}(t)$.

2.8 Velocity and acceleration

Let a particle P move on a trajectory $\mathcal{T}$. Let its position vectors at times t and $(t + \Delta t)$ be $\underline{r}(t)$ and $\underline{r}(t + \Delta t)$. The **velocity** of the particle relative to the frame F is defined as

$$\underline{v} = \lim_{\Delta t \to 0} \frac{\underline{r}(t + \Delta t) - \underline{r}(t)}{\Delta t} = \frac{d\underline{r}}{dt}$$

i.e. as the rate of change of the position vector wrt time. In components

$$\underline{v} = \frac{d\underline{r}}{dt} = \left(\underline{i}\frac{dx}{dt} + \underline{j}\frac{dy}{dt} + \underline{k}\frac{dz}{dt}\right) = (\dot{x}, \dot{y}, \dot{z})$$

or

$$\underline{v} = (u, v, w) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

The velocity vector $\underline{v}$ is tangent to the trajectory.

Similarly the **acceleration** of the particle P is defined as the rate of change velocity wrt t

$$\underline{a} = \frac{d\underline{v}}{dt} = \left(\underline{i}\frac{d^2x}{dt^2} + \underline{j}\frac{d^2y}{dt^2} + \underline{k}\frac{d^2z}{dt^2}\right) = (\ddot{x}, \ddot{y}, \ddot{z})$$

or

$$\underline{a} = (a_x, a_y, a_z) = \left(\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}\right) \equiv \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right)$$

Example: The position vector of a particle moving in the xy plane is given by

$$\underline{r} = a \sin \omega t \underline{i} + a \cos \omega t \underline{j}$$

where a and ω are constants. Show that the particle moves on a circular trajectory and calculate its velocity and acceleration.

Solution: The components of the position vector are $x(t) = a \sin \omega t$, $y(t) = a \cos \omega t$. Treating this as parametric Eq of trajectory, can obtain Eq by eliminating time. Squaring and adding:

$$x^2 + y^2 = a^2(\sin^2 \omega t + \cos^2 \omega t) = a^2$$

which is Eq of a circle of radius a centred at origin.

$$\begin{aligned} \underline{v} &= \frac{d\underline{r}}{dt} = a\omega \cos \omega t \underline{i} - a\omega \sin \omega t \underline{j} \\ \underline{a} &= \frac{d\underline{v}}{dt} = -a\omega^2 \sin \omega t \underline{i} - a\omega^2 \cos \omega t \underline{j} = -\omega^2 \underline{r} \end{aligned}$$

which shows that acceleration is in opposite direction to the position vector pointing towards the origin.

2.9 Unit vectors in polar coordinates

Similar to unit vectors in Cartesian coords, we can associate unit vectors to r and θ in polar coordinates, denoted by $\hat{e_r}$ and $\hat{e_\theta}$. **Note that unlike $\underline{i}, \underline{j}$ these unit vectors are not fixed and change with time (or the angle θ), as the particle moves.**

From the figure we see that $\underline{r} = r\hat{e_r}$. Further

$$\hat{e_r} = \underline{i}\cos\theta + \underline{j}\sin\theta, \qquad \hat{e_\theta} = -\underline{i}\sin\theta + \underline{j}\cos\theta$$

2.9.1 Vel & accel in terms of $\hat{e_r}$ and $\hat{e_\theta}$

To express vel & accel in terms of $\hat{e_r}$ and $\hat{e_\theta}$ recall that the position vector is $\underline{r} = r\hat{e_r}$. Diff wrt t:

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d(r\hat{e_r})}{dt} = \hat{e_r}\frac{dr}{dt} + r\frac{d\hat{e_r}}{dt} \tag{1}$$

Using expressions from above for $\hat{e_r}$ and $\hat{e_\theta}$ in terms of $\underline{i}$ and $\underline{j}$ and using the chain rule we have:

$$\begin{aligned} \frac{d\hat{e_r}}{dt} &= \frac{d\hat{e_r}}{d\theta}\frac{d\theta}{dt} = \left(-\underline{i}\sin\theta + \underline{j}\cos\theta\right)\frac{d\theta}{dt} = \dot{\theta}\hat{e_\theta} \\ \frac{d\hat{e_\theta}}{dt} &= \frac{d\hat{e_\theta}}{d\theta}\frac{d\theta}{dt} = \left(-\underline{i}\cos\theta - \underline{j}\sin\theta\right)\frac{d\theta}{dt} = -\dot{\theta}\hat{e_r} \end{aligned}$$

Substituting in Eq (1):

$$\underline{v} = \dot{r}\hat{e_r} + r\dot{\theta}\hat{e_\theta}$$

Similarly for the acceleration

$$\begin{aligned} \underline{a} &= \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{e_r} + r\dot{\theta}\hat{e_\theta}\right) \\ &= \ddot{r}\hat{e_r} + \dot{r}\frac{d\hat{e_r}}{dt} + \dot{r}\dot{\theta}\hat{e_\theta} + r\ddot{\theta}\hat{e_\theta} + r\dot{\theta}\frac{d\hat{e_\theta}}{dt} \\ &= \ddot{r}\hat{e_r} + \dot{r}\dot{\theta}\hat{e_\theta} + \dot{r}\dot{\theta}\hat{e_\theta} + r\ddot{\theta}\hat{e_\theta} - r\dot{\theta}^2\hat{e_r} \\ \underline{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e_r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{e_\theta} \end{aligned}$$

2.10 Line integrals

We shall have occasions to integrate vector functions along given paths in space, in the form:

$$\int_C \underline{f} \cdot d\underline{r}$$

where $\underline{r}$ is the position vector of a point on the path.

Consider a case where the function and the position vector are functions of time in the forms

$$\underline{f} = \underline{f}(t), \qquad \underline{r} = \underline{i}x(t) + \underline{j}y(t) + \underline{k}z(t)$$

The above integral can be written as

$$\int_C \underline{f}(t) \cdot \frac{d\underline{r}}{dt} dt$$

and evaluated as usual.

Example: Evaluate the integral $\int_p^q \underline{f} \cdot d\underline{r}$, where $\underline{f} = xy\underline{i} + yz^2\underline{j} + y^2z\underline{k}$ and p and q have coordinates $(0,0,0)$ and $(1,1,1)$ respectively. Calculate this integral along the path given by $\underline{r} = t^2\underline{i} + t^3\underline{j} + t^4\underline{k}$.

Solution: From the expression for the path, we have $x = t^2, y = t^3, z = t^4$. Substituting for x, y, z in terms of t in $\underline{f}$:

$$\underline{f} = \underline{i}t^5 + \underline{j}t^{11} + \underline{k}t^{10}$$

and recalling

$$\frac{d\underline{r}}{dt} = \frac{d}{dt}\left(\underline{i}t^2 + \underline{j}t^3 + \underline{k}t^4\right) = 2t\underline{i} + 3t^2\underline{j} + 4t^3\underline{k}$$

Thus

$$\underline{f}(t) \cdot \frac{d\underline{r}}{dt} = 2t^6 + 3t^{13} + 4t^{13}$$

Now at p, $t = 0$ and at q, $t = 1$. Thus

$$\int_p^q \underline{f} \cdot d\underline{r} = \int_0^1 \left(2t^6 + 7t^{13}\right) dt = \frac{2}{7} + \frac{1}{2} = \frac{11}{14}$$

2.11 Gradient and Curl

Gradient: An important differential operator (the dell operator), defined as

$$\nabla = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}.$$

Consider a scalar function $\phi = \phi(x, y, z)$. Acting with ∇ on ϕ gives

$$\nabla\phi = \underline{i}\frac{\partial\phi}{\partial x} + \underline{j}\frac{\partial\phi}{\partial y} + \underline{k}\frac{\partial\phi}{\partial z}$$

which is a vector called the **gradient** of ϕ.

Curl: Another important operator which can act on vector functions is that of **curl** defined as $(\nabla\times)$. Thus the curl of the vector function

$$\underline{a} = \underline{i}a_1(x, y, z) + \underline{j}a_2(x, y, z) + \underline{k}a_3(x, y, z)$$

is given by

$$\nabla \times \underline{a} = \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\right) \times \left(\underline{i}a_1 + \underline{j}a_2 + \underline{k}a_3\right)$$

or

$$\nabla \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Example: Calculate the gradient of the scalar function $\phi = xy^2 z$

$$\begin{aligned} \nabla\phi &= \underline{i}\frac{\partial(xy^2 z)}{\partial x} + \underline{j}\frac{\partial(xy^2 z)}{\partial y} + \underline{k}\frac{\partial(xy^2 z)}{\partial z} \\ &= y^2 z\underline{i} + 2xyz\underline{j} + xy^2\underline{k} \end{aligned}$$

Example: Calculate the curl of the vector function $\underline{a} = xy\underline{i} + \sin y\underline{j} + xe^y\underline{k}$.

$$\nabla \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & \sin y & xe^y \end{vmatrix}$$

$$= (xe^y)\underline{i} - (e^y)\underline{j} - x\underline{k}$$

An important identity

The curl of gradient of a scalar field is zero, i.e.

$$\nabla \times \nabla\phi = 0$$

To see this use the definition of curl and grad:

$$\nabla \times \nabla\phi = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right)\underline{i}$$

$$+ \left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right)\underline{j} + \left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)\underline{k}$$

$$= 0$$

assuming that the scalar function is sufficiently smooth so that the partial derivatives commute.

Thus if $\underline{a} = \nabla\phi$ then its curl is zero ($\nabla \times \underline{a} = 0$). In this case we say the vector field $\underline{a}$ has a scalar potential ϕ associated with it.

Importantly the converse (not proved here) is also true, i.e. **if $\nabla \times \underline{a} = 0$, then there exists a scalar function (called potential) ϕ such that $\underline{a} = \nabla\phi$.**

3 Kinematics

Study of motion without reference to forces.

Given velocity or acceleration can integrate (once or twice) to find $\underline{r}$. Similarly given $\underline{r}$ can differentiate to find velocity of acceleration.

Below we consider some simple examples

3.1 Motion in 1D

In this case we can, wlog, assume motion is along the x-axis. So we drop the underlines and have

$$x = x(t), \qquad v = \frac{dx}{dt}, \qquad a = \frac{d^2x}{dt^2}$$

Example: A particle moving along the x-axis has velocity $v = 3t^2 - 2t + 3$. Calculate its initial acceleration (at $t = 0$) and its displacement when $t = 2$.

We are given that $x = 5$ when $t = 1$.
Diff: $a = \frac{dv}{dt} = 6t - 2$ which at $t = 0$ is -2.
Integrating: $x = \int v dt = t^3 - t^2 + 3t + c$ which using $x = 5$ at $t = 2$ gives $c = 2$.

Motion with constant acceleration:

Consider motion under const accel , a, in 1D, where at $t = 0$, $v(0) = v_0$ and $x(0) = x_0$.

Using the chain rule we have

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{d}{dx}(v^2) = a$$

Integrating wrt x gives $v^2 = {v_0}^2 + 2a(x - x_0)$

A very useful relation in obtaining acceleration if we know initial and final velocities and distance covered.

Example: In a motor cycle race the competitors have to start and finish at rest ten seconds after starting. The winner is the person who has covered the greatest distance. There are 3 competitors: Hope, Dawn and Honesty. Hope's bike accelerates at $2m/sec^2$ and decelerates at $8m/sec^2$, Dawn's accelerates at $3m/sec^2$ and decelerates at $3m/sec^2$ and Honesty's accelerates at $4m/sec^2$ and decelerates at $1m/sec^2$. Who will win the race and how far had she/he gone in 10 seconds?

3.2 Motion in 3D

Example: The position vector of a particle is given by

$$\underline{r} = b\cos\omega t\underline{i} + b\sin\omega t\underline{j} + ct\underline{k}$$

where b, c, ω are constants.

This implies that position of particle at t is given by

$$x = b\cos\omega t, \qquad y = b\sin\omega t, \qquad z = ct$$

Then

$$x^2 + y^2 = b^2$$

which implies that the particles moves on a circular cylinder with OZ as axis

3.3 Laws of Newton

Allow calculation of how velocities/distances change when forces are applied. Before stating these laws, we require a number of definitions.

Inertial frames (IF): Recall that that there are an ∞ number of frames (such as accelerating ones) relative to which motion would look different.

Newton measured motion relative to a preferred subset of frames referred to as **Inertial frames.** *These are frames relative to which an isolated, non-rotating unaccelerated body moves on a straight line and uniformly.*

Newton defined such frames in terms of absolute space (an unsatisfactory notion) which he identified with the centre of mass of the solar system and the fixed stars.

In practice use approximate IFs, such as laboratory, Sun/fixed stars etc. These are satisfactory as in these cases acceleration of frame $\ll$ accel of interest.

Definition: For a body of mass m moving with velocity $\underline{v}$, the **momentum** is defined as $m\underline{v}$.

First law: Any material body continues in its state of rest or uniform motion (in a straight line) unless it is made to change that state by forces acting on it. [Equivalent to statement of existence of IFs]

Second law: Force $\underline{F}$ acting on a point mass m induces an acceleration $\underline{a}$ which is equal to the rate of change of momentum of the mass m

$$\underline{F} = \frac{d(m\underline{v})}{dt}$$

which for constant m **becomes**,

$$\begin{aligned} \underline{F} &= m\frac{d\underline{v}}{dt} = m\underline{a} \\ &= m\left(\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}\right) = m\left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right) \end{aligned}$$

Note: given a force law, $\underline{F}$, this is a 2nd order ODE

Third law: For every action there is an equal and opposite reaction.

3.4 Newtonian framework

Above laws together with the following assumptions, amount to the Newtonian framework:

(i) Space and time are continuous (not discrete). Necessary to allow the employment of calculus.

(ii) There is a universal (absolute) time, i.e. different observers in different frames measure the same time. [Newton also took space to be absolute, but this is not necessary for the Newtonian framework].

(iii) Mass invariant as viewed from different IF's.

(iv) Geometry of space is Euclidean, which for e.g. implies that the sum of angles in any triangle equals 180 degrees.

(v) There is no limit to the accuracy with which time, space intervals and velocity can be measured.

It turns out (ii) & (iii) are relaxed in Special Relativity, (iv) is relaxed in General Relativity & (v) is relaxed in Quantum Mechanics.

3.5 Universal law of gravitation:

In order to account for prior accurate observations of planetary orbits Newton postulated (in his book 'Naturalis Principia Mathematica' ("the Principia"), 1687) that the force between 2 **point particles** of masses m_1 and m_2 a distance r apart is

$$\underline{F} = -\frac{G m_1 m_2}{r^2}\hat{r}$$

where $\hat{r} = \underline{r}/r$ is a unit vector in the direction of $\underline{r}$.

Acceleration due to gravity near the surface of the earth:

Consider a particle of mass m a distance h above the surface of the earth. The force on a particle due to the gravitational force of the Earth is

$$-\frac{Gmm_e}{r^2}\hat{r} = -\frac{Gmm_e}{(R_e + h)^2}\hat{r}$$

where m_e and R_e are the mass and the radius of the Earth.

If $h \ll R_e$ (particle near earth) we can ignore h in the denominator and the acceleration due to gravity, g, is this force divided by m i.e.

$$-\underline{g} = -\frac{Gm_e}{{R_e}^2}\hat{r}, \quad g = \frac{Gm_e}{{R_e}^2} = const$$

Thus the acceleration due to gravity near surface of the Earth is well approximated by a constant $g = 9.8m/sec^2$.

3.6 Units and dimensions in brief

Usually use SI units.

- Length (L, meter, m),
- Time (T, second, s).
- Mass (M, kilogramme, kg)

Other quantities can be derived from these.

- Velocity ($\frac{L}{T}$, meter/second, m/s),
- Acceleration ($\frac{L}{T^2}$, meter/sec squared, m/s^2),
- Force ($\frac{ML}{T^2}$, kg.meter/second^2, mkg/s^2),
- Momentum ($\frac{ML}{T}$, mkg/s) etc.

Some useful constants:

Speed of light: $c = 3 \times 10^8\ m/s$
Radius of the earth: $R = 6378.1 \times 10^3 m$
Mass of the Earth: $M_e = 5.9742 \times 10^{24}\ kg$
Acceleration due to gravity: $g = 9.8 m/s^2$
Universal cons of grav G$= 6.67 \times 10^{-11}\ m^3/kgs^2$.
Escape velocity from Earth $11 \times 10^3 km/s$

3.7 Applications of Newton's 2nd law

Problem reduces to specifying the force $\underline{F}$ and solving the differential equations of motion

$$m\frac{d^2\underline{r}}{dt^2} = \underline{F}$$

which **in 3D are three 2nd order ODEs.**

3.7.1 Motion of particles in 1D

In this case the force is a scalar (has only 1 component. e.g. for motion in x direction $F = F(x)$).

Following classes of problems can arise in 1D, depending on nature of force:

1. $F = cons$ or function of t only
2. $F = F(x)$
3. $F = F(v)$
4. $F = F(x, v)$
5. $F = F(x, v, t)$

3.7.2 Motion in 1D under a force $F = \text{const}$

Simplest example that of motion under a constant force, F, acting on a mass m which moves in direction of force (say along x-axis)

Eq. of motion: $m\frac{d^2x}{dt^2} = F$. Integrating twice wrt t and using initial conditions $x = x_0$, $\frac{dx}{dt} = v_0$ when $t = 0$:

$$\frac{dx}{dt} = \frac{F}{m}t + C$$

and

$$x = \frac{F}{2m}t^2 + Ct + D$$

where C and D are arbitrary constants of integration. Using the initial conditions we find $C = v_0$ and $D = x_0$ and the solution becomes

$$x = \frac{1}{2}\left(\frac{F}{m}\right)t^2 + v_0t + x_0$$

3.7.3 Motion in 1D under gravity: $F = F(r)$

Consider motion of a particle of mass m thrown vertically upwards near the surface of the Earth, mass M_e, radius R_e.

The equation of motion in the vertical direction reduces to 1D motion (i.e. the radial component only):

$$m\frac{d^2r}{dt^2} = -\frac{GM_e m}{r^2}$$

where r is measured from the centre of the earth.

To solve multiply both sides of Eq by $\frac{dr}{dt}$ & divide by m

$$\frac{dr}{dt}\frac{d^2r}{dt^2} = -\frac{GM_e}{r^2}\frac{dr}{dt}$$

Rewrite as

$$\frac{d}{dt}\left(\frac{1}{2}\left(\frac{dr}{dt}\right)^2\right) = \frac{d}{dt}\left(\frac{GM_e}{r}\right)$$

which can be easily integrated to give

$$\left(\frac{dr}{dt}\right)^2 = v^2 = \frac{2GM_e}{r} + \text{constant}$$

If initially at $r = R_e$ velocity is u_0 then

$$\left(v^2 - {u_0}^2\right) = 2GM_e\left(\frac{1}{r} - \frac{1}{R_e}\right)$$

Three possibilities arise:

- **If** ${u_0}^2 > \frac{2GM_e}{R_e}$ then v^2 is positive as $r \to \infty$ and the mass completely escapes from earth.

- **If** $u^2 < \frac{2GM_e}{R_e}$ the velocity v becomes zero when

  $$r = \frac{R_e}{1 - R_e u^2/2GM_e}$$

  and the mass falls back on to earth.

- **If** $u^2 = \frac{2GM}{R}$ particle just escapes to ∞. This velocity

  $$v_E = \left(\frac{2GM_e}{R_e}\right)^{1/2}$$

  is called the **critical or escape** velocity ($= 11km/se$).

Black holes

Consider the escape velocity for a star of mass M and radius R

$$v_E = \left(\frac{2GM}{R}\right)^{1/2}$$

If M remains fixed but R decreases (as in the case of a collapsing star), then the escape velocity will increase.

As R decreases there comes a time when v_E becomes equal to the velocity of light, c, i.e.

$$\left(\frac{2GM}{R}\right) = c^2$$

If the radius is decreased still further, then even light will not be able to escape such a body. Such a body would therefore emit no light (appear black) and is known as a **black hole**.

Historically Black holes were hinted at by Laplace in 1795. He, however, assumed that light is made of particles which obey $\underline{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$, as if they were classical test particles. But they are not.

One would need Einstein's theory of General Relativity to do this properly, but interestingly the result turns out to be the same!

Journey to the Moon

Jules Verne in his book 'A journey to the Moon' suggests sending people to the Moon by placing them in a cannon ball and firing it to the Moon.

According to him a starting velocity of $\sim$ 16km/sec which is reduced by air drag to $\sim$ 11 km/sec, is quite sufficient to carry the ball to the Moon. He assumes the length of the barrel to be 210 m.

What are your objections to the project?

How would you modify the length of the barrel to give the travellers a comfortable ride? (the acceleration due to gravity is 9.8 m/sec^2 and you may assume constant acceleration in the barrel).

3.7.4 Motion in 1D under a resistive force $F = F(v)$

In reality motion is effected by resistive forces such as air resistance or friction. In general such forces depend on the magnitude of the velocity. In particular for slow motion $F(v) \propto v$ and for fast motion $F(v) \propto v^2$. So a good approximation for such forces is found to be $F \propto v^n$, where n depends on the velocity.

Example: Consider the motion of a particle along the x-axis subject only to a resistive force $F(v) = -kv$, where k is a positive const with initial velocity $v = v_0$ at $t = 0$.

Eq. of motion

$$F = ma \equiv m\frac{dv}{dt} = -kv$$

Integrating

$$\int_{v_0}^{v} \frac{dv}{v} = \int_0^t -\frac{k}{m}dt$$

$$[\ln v]_{v_0}^{v} = \left[-\frac{k}{m}t\right]_0^t \Longrightarrow \ln\frac{v}{v_0} = -\frac{k}{m}t$$

$$v = v_0 e^{\left(-\frac{k}{m}t\right)}$$

Thus as $t \to \infty$, $v \to 0$, i.e. it takes an infinite time for the particle to come to rest.

Example: Consider the motion of a particle along the x-axis subject to a constant force F plus a resistive force $F(v) = -kv$, i.e. a total force of $F(v) = F - kv$.

The Eq. of motion is:

$$m\frac{d^2x}{dt^2} \equiv m\frac{dv}{dt} = F - kv$$

Solving for v gives

$$\int \frac{dv}{F - kv} = \frac{1}{m}\int dt$$

$$v = \frac{F}{k}\left(1 - Ae^{-\frac{k}{m}t}\right)$$

where A is a constant of integration.

Thus as opposed to the previous case, as $t \rightarrow \infty$, $v \rightarrow F/k$, i.e. the velocity does not go to zero but reaches a **limiting velocity**.

Falling raindrops

Forces in this case are gravity and air resistance, which for large speeds can be approximated by $F \propto v^2$

The Eq. of motion of raindrop of mass m is

$$m\frac{dv}{dt} = mg - kv^2$$

where k is a constant. Separating variables we have

$$\int \frac{dv}{1 - \left(\frac{k}{mg}\right) v^2} = \int g dt$$

Letting $n^2 = k/mg$ and factorising the integrand gives

$$\int \frac{dv}{1 - nv} + \int \frac{dv}{1 + nv} = 2gt + \text{const}$$

Hence

$$\frac{1}{n}\ln\left(\frac{1+nv}{1-nv}\right) = 2gt + \text{const}$$

Rearranging gives

$$v = \frac{1}{n}\ln\left(\frac{e^{2ngt}-1}{e^{2ngt}+1}\right) = \frac{1}{n}\tanh ngt$$

Thus recalling that

$$\lim_{t\to\infty}\tanh(ngt) \to 1$$

we find that as

$$t \to \infty, \qquad v \to \frac{1}{n}$$

i.e. v approaches a **terminal velocity** v_0**.**

3.7.5 Motion of particles in 2D: projectiles

Consider a mass m fired at an angle θ to the horizontal with velocity u_0 from the surface of the Earth.

Eq of motion has 2 components, in x and y directions:

$$
\begin{aligned}
m\frac{d^2x}{dt^2} &= 0 \\
m\frac{d^2y}{dt^2} &= -mg
\end{aligned}
$$

Initially, at $t = 0$, the horizontal and vertical velocities are given by

$$\left|\frac{dx}{dt}\right|_{t=0} = u_0 \cos\theta, \qquad \left|\frac{dy}{dt}\right|_{t=0} = u_0 \sin\theta$$

Hence integrating the Eqs of motion:

$$\begin{aligned} x &= u_0 \cos\theta t \\ y &= u_0 \sin\theta t - \frac{1}{2} g t^2 \end{aligned}$$

The mass returns to ground when $y = 0$, i.e. after time $t = 2u_0 \sin\theta / g$. The value of x at this time is called **the range**, R, where

$$R = u_0 \cos\theta t = \frac{2{u_0}^2 \sin\theta \cos\theta}{g} = \frac{{u_0}^2}{g} \sin 2\theta$$

For a given initial velocity u_0, the **maximum value of R is when $\sin 2\theta = 1$, i.e. $\theta = \pi/4$ and the range in ${u_0}^2/g$.**

3.8 Consequences of Newton's Laws

Recall the second law is a 2nd order ODE. To solve the Eq. of motion we need to integrate twice.

Interesting information can be obtained by integrating the 2nd law once, wrt to time and x respectively.

3.8.1 Impulse and momentum

Consider a particle of mass m moving under the action of force $\underline{F}$.

Then according to the 2nd law: $\underline{F} = m\underline{a} = m\frac{d\underline{v}}{dt}$.

Integrating with respect to t:

$$\int_{t_1}^{t_2} \underline{F} dt = \int_{t_1}^{t_2} m \frac{d\underline{v}}{dt} dt \qquad (2)$$

$$= \int_{\underline{v}_1}^{\underline{v}_2} m d\underline{v} = m\underline{v}_2 - m\underline{v}_1 \qquad (3)$$

where $m\underline{v}$ is the **momentum** of the particle.

Conservation of momentum: if the total force acting on a particle is zero (i.e. LHS of (2)), then $m\underline{v}_2 = m\underline{v}_1$ i.e. **momentum is conserved**.

3.8.2 Work and Kinetic Energy

Integrating with respect to $\underline{r}$ & using chain rule

$$\begin{aligned}
\int_{\underline{r}_1}^{\underline{r}_2} \underline{F}\cdot d\underline{r} &= = \int_{\underline{r}_1}^{\underline{r}_2} m\frac{d\underline{v}}{dt}\cdot\frac{d\underline{r}}{dt}dt = \int_{\underline{r}_1}^{\underline{r}_2} m\frac{d\underline{v}}{dt}\cdot\underline{v}dt \\
&= \frac{1}{2}m\int_{\underline{v}_1}^{\underline{v}_2} \frac{d}{dt}\left(\underline{v}\cdot\underline{v}\right)dt = \left[\frac{1}{2}m\left|\underline{v}\right|^2\right]_{\underline{v}_1}^{\underline{v}_2} \\
&= \frac{1}{2}m\left|\underline{v}_2\right|^2 - \frac{1}{2}m\left|\underline{v}_1\right|^2
\end{aligned}$$

$\frac{1}{2}m\left|\underline{v}\right|^2$ is the **kinetic energy** (KE) of the particle

$\int_{\underline{r}_1}^{\underline{r}_2} \underline{F}\cdot d\underline{r}$ is defined as the **work** done by the external force $\underline{F}$ on the particle as it moves from $\underline{r}_1$ to $\underline{r}_2$.

Thus total work done by the force on particle (LHS) is equal to the change in its KE (RHS).

3.8.3 Conservative forces

In general (3D) the work done is defined as the line integral

$$\int_{\underline{r}_1}^{\underline{r}_2} \underline{F} \cdot d\underline{r} \tag{4}$$

where $\underline{r}$ is the position vector of a particle as it moves from point $\underline{r}_1$ to $\underline{r}_2$.

<u>Conservative forces:</u> forces for which the work done (4) does not depend on the path taken by the particle as it moves from point $\underline{r}_1$ to $\underline{r}_2$.

An immediate consequence of this is that the work done around a closed loop is zero, i.e.

$$\oint_C \underline{F} \cdot d\underline{r} = 0, \tag{5}$$

which gives a condition for force to be conservative.

The condition for the force to be conservative can alternatively be expressed as

$$\nabla \times \underline{F} = 0, \tag{6}$$

i.e. that the curl of the force should be zero.

Alternatively the force could be expressible as the gradient of a scalar potential thus

$$\underline{F} = -\nabla\Phi \tag{7}$$

where the negative sign is chosen as a convention.

Thus each of the conditions (5), (6) and (7) ensure that the force is conservative.

To see how these definitions are equivalent note that recalling the identity $\nabla \times \nabla\Phi = 0$ it is clear that (7) would imply (6).

Also (6) implies (5) by recalling the Stokes' theorem according to which

$$\int_S (\nabla \times \underline{F}) \cdot d\underline{a} = \oint_C \underline{F} \cdot d\underline{r}$$

where C is a simply connected closed path and S is the surface enclosed by the boundary C. Thus if $\nabla \times \underline{F} = 0$ then the LHS is zero which would make the RHS zero and hence (5).

Similarly given (7) then substituting in the definition of work above

$$\int_{\underline{p}_1}^{\underline{p}_2} \underline{F} \cdot d\underline{r} = -\int_{\underline{p}_1}^{\underline{p}_2} \nabla\Phi \cdot d\underline{r}$$

$$-\int_{\underline{p}_1}^{\underline{p}_2} \left(\underline{i}\frac{\partial\Phi}{\partial x} + \underline{j}\frac{\partial\Phi}{\partial y} + \underline{k}\frac{\partial\Phi}{\partial z}\right) \cdot \left(\underline{i}dx + \underline{j}dy + \underline{k}dz\right)$$

$$-\int_{\underline{p}_1}^{\underline{p}_2} \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz = -\int_{\underline{p}_1}^{\underline{p}_2} d\Phi = -\left[\Phi\right]_{\underline{p}_1}^{\underline{p}_2}$$

Similarly

$$\oint \underline{F} \cdot d\underline{r} = \oint d\Phi = 0$$

which shows that the integrand is a total derivative of a scalar function and therefore only dependent on its end points and not the path, as required for a conservative force.

To summarise: for a force $\underline{F}$ to be conservative it is sufficient for it to satisfy 1 of the conditions:

$$\oint_C \underline{F} \cdot d\underline{r} = 0, \quad \nabla \times \underline{F} = 0, \quad \text{or} \quad \underline{F} = -\nabla\Phi$$

In 1D the situation is very simple and we have:

Theorem: Forces of type $F = F(x)$ in one dimension are conservative.

Proof: Trivial as in 1D path is fixed and integral in going round a closed path reduces to just going (from point x_1 to x_2 say) and coming back, i.e.

$$\int_{x_1}^{x_2} F dx + \int_{x_2}^{x_1} F dx = 0$$

which is true since $\int_{x_1}^{x_2} F dx = -\int_{x_2}^{x_1} F dx$.

3.8.4 Potential energy

Existence of conservative forces motivates the introduction of teh concept of **potential energy** (PE).

Let Φ be a differentiable function of (x, y, z) in Cartesian (or of (r, θ, ϕ) in spherical polar) coordinates, defined s.t the force can be expressed in terms of Φ as

$$\underline{F} = -\nabla\Phi$$

Then as was shown above

$$\int_{\underline{p}_1}^{\underline{p}_2} \underline{F} \cdot d\underline{r} = -\left[\Phi\right]_{\underline{p}_1}^{\underline{p}_2}$$

Defined thus Φ is called the **potential energy** corresponding to the force $\underline{F}$. The $-$ve sign is a convention to signify that $+\Phi$ is the work done against the force and not by it - like a spring that stores PE when compressed.

In 1D we would have $\Phi = \Phi(x)$ such that

$$F = -\frac{d\Phi}{dx}, \qquad \text{and} \qquad \Phi = -\int F dx$$

where the constant of integration is taken as zero

Example: Calculate the potential in 1D corresponding to the force $F(x) = 2x + 3x^2 + 4x^3$.

$$\Phi = -\int F(x)dx = -x^2 - x^3 - x^4$$

Remember the $-$ve sign & the fact that constant of integration is always taken as zero.

Example: Show that the gravitational force is conservative, with the corresponding potential

$$\Phi(r) = -\frac{Gm_1 m_2}{r}, \qquad r \equiv |\underline{r}|$$

Solution: To see note:

$$\Phi(r) = -\int \underline{F} \cdot d\underline{r} = \int \frac{Gm_1 m_2}{r^3}\underline{r} \cdot d\underline{r}.$$

But

$$\underline{r} \cdot d\underline{r} = \frac{1}{2}d(\underline{r} \cdot \underline{r}) = \frac{1}{2}d(r^2) = rdr$$

Hence

$$-\int \underline{F}\cdot d\underline{r} = \int \frac{Gm_1m_2}{r^3}(rdr) = -\frac{Gm_1m_2}{r}$$

Alternatively can show that $\nabla \times \underline{F} = 0$.

$$\nabla \times \left(-\frac{Gm_1m_2}{r^3}\underline{r}\right) = -Gm_1m_2\nabla \times \left(\frac{\underline{r}}{r^3}\right)$$

$$\begin{aligned} &= -Gm_1m_2 \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix} \\ &= 0 \end{aligned}$$

by calculating directly using

$$r^3 = \left(x^2 + y^2 + z^2\right)^{3/2}$$

Note this shows the force is conservative but does not give the potential!

3.8.5 Conservation of energy

Equating the two expressions for the work done we have

$$\int_{\underline{r}_1}^{\underline{r}_2} \underline{F} \cdot d\underline{r} = \frac{1}{2} m \left|\underline{v}_2\right|^2 - \frac{1}{2} m \left|\underline{v}_1\right|^2 = \Phi(\underline{r}_1) - \Phi(\underline{r}_2)$$

which implies

$$\frac{1}{2} m \left|\underline{v}_1\right|^2 + \Phi(\underline{r}_1) = \frac{1}{2} m \left|\underline{v}_2\right|^2 + \Phi(\underline{r}_2)$$

But LHS depends on point $\underline{r}_1$ and RHS on point $\underline{r}_2$.

Now since these points are arbitrary, this is only possible if both sides are the same constant.

Thus **for a conservative force** we have

$$\frac{1}{2} m \left|\underline{v}\right|^2 + \Phi = KE + PE = E \quad \text{a constant} \qquad (8)$$

E is called the **total energy** of the particle.

In 1D this Eq becomes

$$\frac{1}{2} m\dot{x}^2 + \Phi(x) = E, \qquad \dot{x} \equiv \frac{dx}{dt} = u \qquad (9)$$

3.8.6 Qualitative information about motion using the energy equation: 1D case

The energy conservation law can be used to obtain qualitative information about motion without solving the 2nd law directly.

To see consider motion along x-axis with vel $u = \dot{x}$.

The energy conservation Eq (9) can be written as

$$u \equiv \dot{x} = \pm\sqrt{\frac{2}{m}\left(E - \Phi(x)\right)} \tag{10}$$

Thus for a particle moving subject to force whose PE is $\Phi(x)$ motion is only possible if

$$\Phi(x) \leq E \tag{11}$$

in order to ensure that $\dot{x}$ is real. Thus given a $\Phi(x)$ (or force law $F(x)$ - in which case $\Phi = -\int F(x)dx$) plot $\Phi(x)$ versus x and determine where motion exists (i.e. where $\Phi(x) \leq E$).

Note: the shape of the potential $\Phi(x)$ can tell us a lot about nature of possible motion.

Example: A particle moves under the action of a conservative force whose PE function is given by the graph below. Discuss the possible types of motion that can occur.

Types of motion depends on size of E. For the shown E, motion possible for $x_1 \leq x \leq x_2, \quad x > x_3$.

When particle is at x_A

$$d\Phi/dx < 0 \rightarrow F = -d\Phi/dx > 0$$

i.e. it experiences a force away from the origin.

When placed at x_B

$$F = -d\Phi/dx < 0$$

i.e. it experiences a force towards the origin.

particle comes to rest at x_A, x_B.

So particle oscillates between these points!

When particle at $x > x_3$ force away from origin and particle accelerates to $+\infty$

If particle placed at $x > x_3$ and fired towards origin with energy E, it slows down as it moves towards x_3, stops there and rolls back to $+\infty$ again.

Now change E and repeat analysis.

3.8.7 Equilibrium points

Important for further understanding the dynamics.

Points of equilibrium defined as points at which

$$\frac{d\Phi(x)}{dx} = 0$$

i.e. the turning points of the potential function.

Point of stable equilibrium: point at which $\Phi(x)$ is a minimum (i.e. $d\Phi/dx = 0, \;\; d^2\Phi/dx^2 > 0$).

Stable as motion in their nbhd is oscillatory: small displacements do not results in large divergences.

Point of unstable equilibrium: point at which $\Phi(x)$ is a maximum (i.e. $d\Phi/dx = 0, \;\; d^2\Phi/dx^2 < 0$).

Unstable as small displacements do results in large divergences.

Steps to follow in doing problems

- Given a force law $F(x)$, use $\Phi = -\int F(x)dx$ to calculate the potential.

- Find points of equilibrium of potential, determining whether they are stable or unstable.

- Plot the potential $\Phi(x)$ vs x

- Take different values of energy E to cover all qualitative types of motion that can occur. To decide which values of E to take, look at the shape of the potential.

Example: A particle of unit mass moves along the x-axis under the influence of a force whose magnitude is given by

$$f(x) = -8x + 2x^3.$$

(a) Find an expression for the potential energy $\Phi(x)$. (You may assume $\Phi(0) = 0$ for simplicity.)

(b) Find the points of equilibrium and determine the stability of each.

(c) Sketch $\Phi(x)$ as a function of x and discuss briefly the possible types of motion that can occur depending upon the total energy of the particle.

(d) The particle is placed at the position of stable equilibrium and given a velocity of magnitude u_0 in the positive x direction. Find u_0 such the particle cannot escape to $+\infty$. For what values of u_0 will the particle oscillate about the origin?

4 Motion near a point of stable equilibrium: SHM

We have seen from qualitative analysis that such a motion is oscillatory. Here make this quantitative.

Consider a particle of mass m moving under the action of a force whose PE is $\Phi(x)$.

Without loss of generality choose coordinates such that pt of stable equilibrium of Φ is at origin $x = 0$.

Place the particle at the origin and displace it a little (give it a small KE).

To write Eq. of motion obtain force from

$$F(x) = -\frac{d\Phi(x)}{dx} \tag{12}$$

Since we are interested in motion in nhbhs of origin, use Maclaurin's series to expand $\Phi(x)$:

$$\Phi(x) = \Phi(0) + \frac{x}{1!}\Phi^{'}(0) + \frac{x^2}{2!}\Phi^{''}(0) + ...O(x^3)$$

where $(')= d/dx$. For small enough oscillations, ignore terms of $O(x^3)$.

Recall also that $\Phi(x)$ passes through the origin (i.e. $\Phi(0) = 0$) and the origin is a point of stable equilibrium (i.e. $\Phi^{'}(0) = 0$).

Substituting gives

$$\Phi(x) = \frac{x^2}{2!}\Phi^{''}(0)$$

Letting $\Phi^{''}(0) = k > 0$ we have

$$\Phi(x) = k\frac{x^2}{2!}$$

Now using (12) we obtain $F = -kx$ and the Eq. of motion ($F = m\ddot{x}$) becomes

$$m\ddot{x} + kx = 0$$

This is the equation of **simple harmonic motion (SHM)**.

This is a **linear homogeneous ODE for which the general solution is of the form** [CHECK!]

$$x = ae^{i\omega t} + be^{-i\omega t}, \qquad \omega^2 = \frac{k}{m}$$

where a and b are are arbitrary constants to be specified by ICs. But x is positive distance and cannot take imaginary values. Therefore the arbitrary constants need to be chosen as complex conjugates thus

$$a = \frac{1}{2}(A - iB), \qquad b = \frac{1}{2}(A + iB)$$

Substituting for a and b in x and recalling that

$$e^{\pm i\omega t} = \cos \omega t \pm i \sin \omega t$$

we can write x alternatively as

$$x = A \cos \omega t + B \sin \omega t,$$

A, B are arbitrary constants to be specified by ICs.

Alternatively this solution can be expressed as

$$x = \alpha \cos(\omega t - \theta)$$

by letting $A = \alpha \cos \theta,\ B = \alpha \sin \theta$ and using the trigonometric identity $\cos(\omega - \theta) = ...$

Fixing arbitrary constants:

In practice the arbitrary constants are specified by using initial conditions.

For example, let at $t = 0$, $x = x_0$ and $u = u_0$.

Substituting into

$$x = A\cos\omega t + B\sin\omega t$$

we find $A = x_0$.

Also differentiating x we have

$$u = \dot{x} = -A\omega\sin\omega t + B\omega\cos\omega t$$

which gives $B = u_0/\omega$.

Thus with these ICs we find

$$x = x_0\cos\omega t + \frac{u_0}{\omega}\sin\omega t$$

Remark: These functions are **periodic with period** $2\pi/\omega$. To check recall that **a function** $f(t)$ **is called periodic with period** T **if** $f(t) = f(t+T)$. For example

$$\sin\omega(t+T) = \sin\omega(t+2\pi/\omega) = \sin(\omega t+2\pi) = \sin\omega t.$$

Some important definitions:

α		amplitude
T	$= \dfrac{2\pi}{\omega}$	period
ω		angular frequency
ν	$= \dfrac{1}{T} = \dfrac{\omega}{2\pi}$	frequency
θ		phase

Thus we have proved that motion of a particle of mass m in the neighbourhood of a point of stable equilibrium of any potential energy function $\Phi(x)$ is periodic (satisfies SHM) with period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{\Phi''(0)}}$$

Example: A particle of unit mass is placed at the position of stable equilibrium of the potential energy function

$$\Phi(x) = \left(\frac{x^2}{2} - \frac{x^4}{4}\right)$$

Find period of its oscillations if slightly disturbed.

Solution: For small disturbances we have SHM with period

$$T = \frac{2\pi}{\sqrt{\Phi''}}, \qquad m = 1$$

Now $\Phi^{'} = \left(x - x^3\right)$ Points of equilibrium given by $\Phi^{'} = 0$ i.e. $x = 0, \pm 1$. Also $\Phi^{''} = (1 - 3x^2)$ which is negative at $x = 0$ implying that this a point of stable equilibrium. Also $\Phi^{''}(0) = 1$ hence the period is

$$T = \frac{2\pi}{\sqrt{\Phi''(0)}} = 2\pi$$

Important remark: If the point of stable equilibrium is at $x = a$, then using Taylor expansion instead we again find the motion is SHM with period

$$T = 2\pi\sqrt{\frac{m}{\Phi''(a)}}$$

5 ODEs in brief

For more details see your Differential Equations notes.

5.1 Linear homogeneous ODE with constant coefficients

Consider homogeneous linear ODEs of the form

$$a_2 \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + a_1 \frac{\mathrm{d}x}{\mathrm{d}t} + a_0\, x = 0 \tag{13}$$

where a_0, a_1, a_2 are real constants.

To find the general solution look for solutions of type $x(t) = \mathrm{e}^{pt}$, substitute in the ODE and solve for p.

In the case of the above 2ndn order Eq. this gives the quadratic characteristic equation

$$a_2\, p^2 + a_1\, p + a_0 = 0 \tag{14}$$

whose solution gives the two possible values for p.
Three possibilities arise depending on roots of Eq. (14).

Case A: the above quadratic equation two real roots, say p_1 and p_2. The general solution to the equation (13) is then given by the linear combination of the solutions in the form

$$x(t) = A\mathrm{e}^{p_1 t} + B\mathrm{e}^{p_2 t}$$

where A and B are arbitrary constants

Case B: the above quadratic equation has a repeated root, p say. The general solution to equation (13) is then given by

$$x(t) = (A + Bt)\mathrm{e}^{pt}$$

where A and B are constants

Case C: the above quadratic equation has two complex conjugate roots of the form $p = g \pm hi$. The general solution to equation (13) is

$$x(t) = \alpha\mathrm{e}^{(g+hi)t} + \beta\mathrm{e}^{(g-hi)t}$$

which α, β are arbitrary constants.

Alternatively this can be written as

$$x(t) = \mathrm{e}^{gt}(A\cos(ht) + B\sin(ht))$$

or

$$x(t) = a\mathrm{e}^{gt}\cos(ht + \theta)$$

where A, B, a, α and β are arbitrary constants,
[CHECK that the three forms are equivalent]

5.2 Linear inhomogeneous ODE with constant coefficients

Consider the case where Eq. (13) has a non-zero RHS thus

$$a_2 \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + a_1 \frac{\mathrm{d}x}{\mathrm{d}t} + a_0 x = f(t) \qquad (15)$$

where a_0, a_1, a_2 are constants and $f(t)$ is a function of t.

The general solution to (15) is then given by

$$x(t) = x_h(t) + x_p(t)$$

where $x_h(t)$ is the general solution of the associated homogeneous equation (13) and $x_p(t)$ is a particular solution of the full inhomogeneous Eq. (15).

The solution $x_p(t)$ can usually be guessed by looking at the the RHS of the Eq. (15).

6 Damped SHM

In practice oscillations are damped by resistance, naturally or artificially.

At low velocities, resistance force $F \propto \dot{x}$.

So in presence of a resistive force, equation of motion of SHM is modified by introducing a frictional force $-\alpha\dot{x}$ thus:

$$F = m\ddot{x} = -kx - \alpha\dot{x}$$

where $\alpha > 0$ to ensure that force is damping instead of forcing. We also take $k > 0$ to ensure oscillatory rather than exponential motion.

Equation of **damped simple harmonic motion** is

$$m\ddot{x} + \alpha\dot{x} + kx = 0$$

This is a linear homogeneous ODE with constant coefficients. To solve look for solutions of type

$$x = e^{pt}, \quad \text{with} \quad \dot{x} = pe^{pt}, \ \ddot{x} = p^2 e^{pt}$$

substituting we find

$$mp^2 + \alpha p + k = 0$$

which has solutions

$$p = -\frac{\alpha}{2m} \pm \sqrt{\left(\frac{\alpha}{2m}\right)^2 - \frac{k}{m}}$$

or

$$p = -\gamma \pm \left(\gamma^2 - {\omega_0}^2\right)^{1/2}, \tag{16}$$

where $\gamma = \frac{\alpha}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$ is the angular frequency of the undamped oscillator.

Three cases arise depending upon the size of α or γ.

6.1 Large damping (large α): $\gamma^2 > {\omega_0}^2$

Both roots (16) are real and negative:

$$\begin{aligned} p_1 \equiv -\gamma_1 &= -\gamma - \left(\gamma^2 - {\omega_0}^2\right)^{1/2} \\ p_2 \equiv -\gamma_2 &= -\gamma + \left(\gamma^2 - {\omega_0}^2\right)^{1/2} \end{aligned}$$

The general solution becomes

$$x = Ae^{-\gamma_1 t} + Be^{-\gamma_2 t}$$

So as t grows, the displacement $x \to 0$, i.e. damped out exponentially.

The **characteristic or e–folding time**, defined as the time in which x is reduced by $1/e$, is given by

$$t_{ch}\gamma_2 \sim 1, \quad \text{i.e.} \quad t_{ch} \sim \frac{1}{\gamma_2}$$

6.2 Small damping (small α): $\gamma^2 < {\omega_0}^2$

The roots of (16) are complex conjugates

$$p = -\gamma \pm i\omega \qquad \text{where} \quad \omega = \sqrt{{\omega_0}^2 - \gamma^2}$$

The general solution is

$$\begin{aligned} x &= \frac{1}{2}Ae^{(i\omega-\gamma)t} + \frac{1}{2}Be^{(-i\omega-\gamma)t} \\ &= e^{-\gamma t}\left[\frac{1}{2}Ae^{i\omega t} + \frac{1}{2}Be^{-i\omega t}\right] \end{aligned}$$

where the factor 1/2 is chosen for convenience.

Now since x needs to be real A, B need to be complex conjugates. Letting $A = ae^{-i\theta},\ B = ae^{+i\theta}$ then

$$\begin{aligned}
\frac{1}{2}Ae^{i\omega t} + \frac{1}{2}Be^{-i\omega t} &= \frac{1}{2}a\left\{e^{i(\omega t-\theta)} + e^{-i(\omega t-\theta)}\right\} \\
&= \frac{1}{2}a\left[2\cos(\omega t-\theta) + 0\right] \\
&= a\cos(\omega t-\theta)
\end{aligned}$$

Thus

$$x = ae^{-\gamma t}\cos(\omega t-\theta) \tag{17}$$

This solution represents **oscillations with decaying amplitudes** $\{ae^{-\gamma t}\}$ **& angular velocity** ω**.**

Ratio of neighbouring amplitudes: that is amplitudes at $t = t$ and $t = t + \pi/\omega$.

Using solution (17)

$$\frac{x_2}{x_1} = \frac{ae^{-\gamma(t+\pi/\omega)}}{ae^{-\gamma t}} \cdot \frac{\cos(\omega t + \pi - \theta)}{\cos(\omega t - \theta)}$$

The 2nd term is equal to -1 so the ratio becomes

$$\frac{x_2}{x_1} = -e^{-\frac{\gamma\pi}{\omega}}$$

which is true for all neighbouring amplitudes

$$\frac{x_{n+1}}{x_n} = -e^{-\frac{\gamma\pi}{\omega}}$$

Negative sign due to fact that neighbouring amplitudes are on opposite sides of the x axis.

Thus the amplitudes of successive oscillations decrease in a geometrical progression.

6.3 Critical damping: $\gamma^2 = {\omega_0}^2$

In this case the 2 roots coincide and are $= -\gamma$.

The solution becomes $x = (A + B)e^{-\gamma t}$ which contains only one arbitrary constant $(A + B)$.

This solution is therefore not general. We need a second solution which we can take as [CHECk]

$$x = te^{-\gamma t}$$

The general solution is then

$$x = (a + bt)e^{-\gamma t}$$

To summarise: all damped oscillations eventually die out.

Remark 1: $\gamma_1 > \gamma > \gamma_2$. For long enough times the critical damping solution falls off faster than the large damping case. In most applications (e.g. reading meters, hydraulic springs etc) it is desirable that once displaced, the mechanism returns quickly & smoothly to its equilibrium position. This is achieved by critical damping!

Remark 2: Mechanical energy of oscillations no longer conserved as there is a dissipation due to damping.

Example: Plot the solution to SHM in the form $\alpha \sin(\omega t - \theta)$ in the $x - \dot{x}$ plane.

What happens in the case of small damping case $\alpha e^{-\gamma t} \sin(\omega t - \theta)$?

Example: A particle of unit mass performing (lightly) damped simple harmonic oscillations satisfies the above equation of motion. The three consecutive positions of its instantaneous rest, relative to an arbitrary origin O, are given by $x = a, x = b, x = c$. Show that the particle ultimately comes to rest at a point whose distance x_0 from O is given by

$$x_0 = \frac{ca - b^2}{a - 2b + c}$$

7 Forced, damped SHM

To maintain damped oscillations we need to force them.

Consider the response of a damped oscillator to a time dependent applied force with Eq. of motion:

$$m\ddot{x} + \alpha\dot{x} + kx = F(t) \tag{18}$$

As an example consider the case where $F(t)$ is a periodic function of the form

$$F(t) = F_1 \cos \omega_1 t$$

where F_1, ω_1 are constants.

Eq (18) is an **inhomogeneous linear ODE** with constant coefficients.

The **general solution** has the form:

$$\text{General Solution} x = x_p(t) + x_h(t)$$

where $x_h(t)$ is the general solution of the corresponding homogeneous equation (with $F(t) = 0$);

and $x_p(t)$ is a particular solution of the the inhomogeneous equation (18).

We have already found $x_h(t)$ in the 3 possible cases.

To find $x_p(t)$, the trick is to look for solutions of the type given by the RHS.

Thus we can look for a solution of the form

$$x = a_1 \cos(\omega_1 t - \theta_1)$$

where a_1, θ_1 are constants which can be determined by substituting in (18). Dividing (18) by m and assuming, without loss of generality, $m = 1$, we obtain

$$\ddot{x} + 2\gamma\dot{x} + {\omega_0}^2 = F_1 \cos \omega_1 t \qquad (19)$$

Now

$$\begin{aligned}\dot{x} &= -a_1\omega_1 \sin(\omega_1 t - \theta_1) \\ \ddot{x} &= -a_1{\omega_1}^2 \cos(\omega_1 t - \theta_1)\end{aligned}$$

Substituting we find

$$\begin{aligned}& a_1{\omega_1}^2 \cos(\omega_1 t - \theta_1) - 2\gamma a_1\omega_1 \sin(\omega_1 t - \theta_1) \\ + \quad & {\omega_0}^2 a_1 \cos(\omega_1 t - \theta_1) = F_1 \cos \omega_1 t\end{aligned}$$

or

$$\begin{aligned}& a_1 \left({\omega_0}^2 - {\omega_1}^2\right) \cos(\omega_1 t - \theta_1) \\ - \quad & 2\gamma a_1\omega_1 \sin(\omega_1 t - \theta_1) = F_1 \cos \omega_1 t\end{aligned}$$

Expanding $\cos(\omega_1 t - \theta_1)$ and $\sin(\omega_1 t - \theta_1)$ we have

$$\begin{aligned} & a_1\left({\omega_0}^2 - {\omega_1}^2\right)\left[\cos\omega_1 t\cos\theta_1 + \sin\omega_1 t\sin\theta_1\right] \\ & -2\gamma a_1\omega_1\sin\omega_1 t\cos\theta_1 + 2\gamma a_1\omega_1\cos\omega_1 t\sin\theta_1 \\ = \; & F_1\cos\omega_1 t \end{aligned}$$

Equating coefficients of $\cos\omega_1 t$ and $\sin\omega_1 t$:

$$\begin{aligned} \left[a_1\left({\omega_0}^2 - {\omega_1}^2\right)\cos\theta_1 + 2\gamma a_1\omega_1\sin\theta_1\right]\cos\omega_1 t & \\ = F_1\cos\omega_1 t & \\ \left[a_1\left({\omega_0}^2 - {\omega_1}^2\right)\sin\theta_1 - 2\gamma a_1\omega_1\cos\theta_1\right] &= 0 \end{aligned} \tag{20}$$

From the 2nd Eq. in (20) we have

$$\tan\theta_1 = \frac{2\gamma a_1\omega_1}{a_1\left({\omega_0}^2 - {\omega_1}^2\right)} = \frac{2\gamma\omega_1}{\left({\omega_0}^2 - {\omega_1}^2\right)} \tag{21}$$

Substituting from 2nd into the 1st Eq. in (20) for $\sin\theta_1$ in terms of $\cos\theta_1$ we have

$$a_1\left({\omega_0}^2 - {\omega_1}^2\right)\cos\theta_1 + \frac{4\gamma^2{a_1}^2{\omega_1}^2\cos\theta_1}{a_1\left({\omega_0}^2 - {\omega_1}^2\right)} = F_1$$

or

$$\cos\theta_1\left[{a_1}^2\left({\omega_0}^2 - {\omega_1}^2\right)^2 + 4\gamma^2{a_1}^2{\omega_1}^2\right] = F_1 a_1\left({\omega_0}^2 - {\omega_1}^2\right)$$

Squaring

$$
\begin{aligned}
& \cos^2\theta_1 \left\{ \left[{a_1}^2 \left({\omega_0}^2 - {\omega_1}^2\right)^2 + 4\gamma^2 {a_1}^2 {\omega_1}^2 \right] \right\}^2 \\
= \quad & {F_1}^2 {a_1}^2 \left({\omega_0}^2 - {\omega_1}^2\right)^2 \qquad (22)
\end{aligned}
$$

But

$$
\cos^2\theta_1 = \frac{1}{1+\tan^2\theta_1} = \frac{\left({\omega_0}^2 - {\omega_1}^2\right)^2}{({\omega_0}^2 - {\omega_1}^2)^2 + 4\gamma^2 {\omega_1}^2}
$$

Substituting in (22) and taking the square root:

$$
a_1 = \frac{F_1}{\left[({\omega_0}^2 - {\omega_1}^2)^2 + 4\gamma^2 {\omega_1}^2\right]^{1/2}} \qquad (23)
$$

Thus

$$
x_p = a_1 \cos(\omega_1 t - \theta_1)
$$

with a_1, θ_1 give by expressions (23) and (21).

The general solution is then given by the sum of this x_p plus an appropriate x_h.

For example in the case of small damping we have

$$x = a_1 \cos(\omega_1 t - \theta_1) + a e^{-\gamma t} \cos(\omega t - \theta)$$

Now in general as $t \to \infty$ the 2nd term tends to zero exponentially and it is called the **transient**. Thus

$$\lim_{t \to \infty} x \longrightarrow a_1 \cos(\omega_1 t - \theta_1)$$

Therefore no matter what the ICs, the oscillations are ultimately governed by the external force with the period of the applied force (ω_1) and not that of the undamped oscillator (ω_0).

7.1 Resonance

Looking closely at the expression

$$a_1 = \frac{F_1}{\left[({\omega_0}^2 - {\omega_1}^2)^2 + 4\gamma^2{\omega_1}^2 \right]^{1/2}}$$

we note that the amplitude a_1 is strongly dependent upon the frequencies ω_0 and ω_1. In particular when $\omega_0 = \omega_1$, i.e. when the system's frequency is equal to the forcing frequency, then

$$a_1 \longrightarrow \frac{F_1}{2\gamma\omega_1}$$

Thus if $\gamma \to 0$ (i.e. small damping) , then a_1 becomes very large and in this case we say the system is in resonance with amplitude

$$a_R = \frac{F_1}{2\gamma\omega_1}$$

In practice we often wish to avoid this situation as in case of construction of ships and bridges etc.

8 Motion under a central force

Central forces are directed towards or away from a fixed point (often taken as the origin) thus:

$$\underline{F} = f(r)\underline{\hat{r}}$$

Central forces, include the force of gravity and the electric force:

Examples: Force of gravity

$$\underline{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

which is attractive, i.e. towards the centre (origin)

or

$$F = r^n\hat{r}$$

which is a repulsive force, i.e. away from the centre (origin)

I. Central forces are conservative.

This can be checked by by noting that

$$\int_{\underline{r}_1}^{\underline{r}_2} \underline{F}\cdot d\underline{r} = \int_{\underline{r}_1}^{\underline{r}_2} f(r)\hat{\underline{r}}\cdot d\underline{r} = \int_{\underline{r}_1}^{\underline{r}_2} f(r)\frac{\underline{r}\cdot d\underline{r}}{r} = \int_{\underline{r}_1}^{\underline{r}_2} f(r)dr$$

where we have used

$$\underline{r}\cdot d\underline{r} = \frac{1}{2}\frac{d}{dt}\left(\underline{r}\cdot\underline{r}\right)dt = \frac{1}{2}\frac{d}{dt}\left(r^2\right)dt = rdr$$

Thus the integral is just dependent on r (is 1D) and therefore depends only on the end points and not on the path taken, showing that central forces are conservative.

Thus as for conservative forces we can define a potential Φ and write the conservation law of energy

$$\frac{1}{2}m\left|\underline{v}\right|^2 + \Phi = KE + PE = E \quad \text{a constant}$$

Definition: Angular momentum $\underline{J}$ of a particle of mass m, velocity $\underline{v}$ about an origin a distance $\underline{r}$ is defined

$$\underline{J} = \underline{r}\times\underline{p} = \underline{r}\times m\underline{v}$$

II. Motion under central forces is planar (2D) and conserves angular momentum.

To check look at the rate of change of angular momentum

$$\begin{aligned} \frac{d\underline{J}}{dt} &= \frac{d}{dt}(m\underline{r} \times \underline{v}) \\ &= (m\underline{v} \times \underline{v} + m\underline{r} \times \underline{a}) \\ &= 0 + \underline{r} \times \underline{F} \\ &= \underline{r} \times f(r)\underline{\hat{r}} = f(r)\underline{r} \times \underline{\hat{r}} = 0 \end{aligned}$$

since $\underline{r}$ and $\underline{\hat{r}}$ are parallel and $\underline{v} \times \underline{v} = 0$.

Thus under a central force the angular momentum $\underline{J}$ is constant and hence conserved.

But $\underline{J}$ = constant implies two things: both magnitude and direction of $\underline{J}$ are constants.

We shall now study the consequences of each in turn.

Direction of $\underline{J}$ is constant:

Consider a particle of mass m with position vector $\underline{r}$ and velocity $\underline{v}$ moving under the action of a central force.

Its angular momentum is then given by $\underline{J} = m\underline{r} \times \underline{v}$.

Now from the definition of vector product $\underline{J} = m\underline{r} \times \underline{v}$ is a vector perpendicular to the plane of $\underline{r}$ and $\underline{v}$.

The constancy of the direction of $\underline{J}$ implies that the plane constituted by $\underline{r}$ and $\underline{v}$ remains the same as the particle moves under the action of the central force.

Hence **the motion under central forces are confined to a plane, i.e. is 2-dimensional.**

Magnitude of $\underline{J}$ is constant:

Consider a planet as it moves a small distance along its orbit, say from the point P with coords (r, θ) to a neighbouring point P' with coords $(r + \Delta r, \theta + \Delta\theta)$.

As the particle moves from P to P', the radius vector sweeps an area given approximately by

$$\Delta A = \frac{1}{2} r (r \Delta\theta)$$

which has the rate of change

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t}$$

which in the limit of $\Delta t \to 0$ gives

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta}$$

Now the RHS can be expressed in terms of the magnitude of the angular momentum vector $\underline{J} = m\underline{r} \times \dot{\underline{r}}$. To see this recall that

$$\dot{\underline{r}} = \dot{r}\hat{e_r} + r\dot{\theta}\hat{e_\theta}.$$

Then

$$J \equiv |\underline{J}| = |m\underline{r} \times \dot{\underline{r}}| = mr^2\dot{\theta}$$

Substituting in the RHS gives

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{J}{2m} = \text{constant}$$

which shows that **for motion under a central force the rate of the area swept by the radius vector is a constant**.

This is **Kepler's 2nd law of planetary motions**.

An important consequence of this law for elliptical orbits is that when planets are nearer to the Sun they must move faster than when they are further away as the area swept is a constant.

8.1 Qualitative information about motion under central forces

We saw central forces are conservative so motion under such forces satisfies the energy conservation law

$$\frac{1}{2}m\left|\underline{v}\right|^2 + \Phi = E$$

which recalling that in polar coords $\underline{v} = \dot{r}\hat{e_r} + r\dot{\theta}\hat{e_\theta}$ and $|\underline{v}|^2 = \dot{r}^2 + r^2\dot{\theta}^2$ gives

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \Phi = E$$

We can use $J = mr^2\dot{\theta}$ to eliminate $\dot{\theta}$ and rewrite this Eq. purely in terms of r, thus:

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\frac{J^2}{(mr^2)^2}\right) + \Phi = E$$

or

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + \Phi = E$$

This is referred to as **Radial Energy Equation**.

Now letting

$$\frac{J^2}{2mr^2} + \Phi = U(r)$$

where $U(r)$ is referred to as the **Effective Potential Energy (EPE)**, allows energy Eq to be written as

$$\frac{1}{2}m\dot{r}^2 + U(r) = E$$

<u>**Note:**</u> **This resembles the energy Eq. in 1D, and again motion is possible only if $E \geq U(r)$.**

Important difference: In this case, however, r is a polar distance so above Energy Eq gives information about motion in 2D.

Example: **Derive qualitative information about motion under the action of the gravitational force**

$$\underline{F} = -\frac{Gm_1m_2}{r^2}\hat{r}.$$

Solution: We start with the energy equation

$$\frac{1}{2}m\dot{r}^2 + U(r) = E, \qquad U(r) = \frac{J^2}{2mr^2} + \Phi$$

We have already calculated Φ for the force of gravity:

$$\Phi = -\frac{GmM}{r}$$

Substituting in U gives

$$U(r) = \frac{J^2}{2mr^2} - \frac{GmM}{r}.$$

Following possibilities arise depending on the value of J

<u>J = 0 :</u> In this case the particle moves in a straight line (like a particle thrown up vertically)

For $E < 0$, shooting particle from origin reaches a distance r_1 and then returns to centre.

For $E = 0$, shooting particle from origin just escapes to infinity. **Escape velocity.**

For $E > 0$, shooting particle from origin escapes to infinity, arriving there with positive velocity.

$\mathbf{J \neq 0}$: In this case

$$U(r) = \frac{J^2}{2mr^2} - \frac{GmM}{r}$$

which letting $\ell = \frac{J^2}{Gm^2M}$ becomes

$$U(r) = GmM\left(\frac{\ell}{2r^2} - \frac{1}{r}\right)$$

Now $U(r) = 0$ at $r = \ell/2$ and has a min at $r = \ell$ with $U(\ell) = -GMm/2\ell$.

For motion to exist we need $E \geq U$.

For $E < -GmM/2\ell$, motion not possible

For $E = -GmM/2\ell$ we have $\dot{r}^2 = 0$, which implies $r =$ constant, which is the Eq of a circle in polar coords. Thus **motion is circular**.

For $-GmM/2\ell < E < 0$, motion is confined to $r_1 \leq r \leq r_2$, i.e. motion is bounded, periodic and r can vary between a min and max distance. Thus **motion is elliptical**.

For $E = 0$. In this case particle can escape to infinity, arriving there with zero speed, i.e. **motion is parabolic**.

For $E > 0$. In this case particle can escape to infinity, arriving there with positive speed, i.e. **motion is hyperbolic**.

9 Orbits

Above using the Energy equation we gave qualitative information about possible types of orbits under the force of gravity.

Here we wish to obtain quantitative information by deriving the equation describing the orbits particles under the force of gravity.

This will also allow us to derive the orbits of planets around the Sun.

An immediate problem with the latter is that Sun and planets are not point particles, but extended nearly spherical bodies.

This raises the question of whether Newton's inverse square law of gravitation between 2 **point particles** still holds in the case of Sun and planets.

This question is answered by the following theorem.

9.1 Newton's sphere theorem

Starting with a spherical body of uniform density, this theorem shows when it can be treat as a point particle.

We proceed in a number of steps:

- Subdivide sphere (assumed to have a uniform density) into a set of concentric thin spherical shells.

- Subdivide each shell into a set of rings all points of which are equi–distant to the point mass outside or inside the shell.

- Calculate the potential due to such a ring. Being a scalar, potential is much easier to calculate than the force.

- Add the potentials due to all rings constituting the shell by simple integration.

- Calculate the force due to shell.

- Add effects of all shells to find force due to whole sphere.

Potential due to a spherical shell:

Consider the potential due to a spherical shell of uniform density and mass M on a particle of mass m placed a distance r_0 from its centre at a point P outside or inside the shell.

Let the shell have a radius a, area $4\pi a^2$ and mass per unit area $\sigma = M/4\pi a^2$.

We first consider a ring defined by a slice through the spherical shell, perpendicular to the line connecting P to the centre of the shell, i.e. a rotating element around P subtending an angle $d\theta$ with thickness $ad\theta$. The radius of the ring is $a\sin\theta$, its area is $(2\pi a\sin\theta)ad\theta$ and its mass is $\sigma(2\pi a\sin\theta)ad\theta$.

The potential due to this ring at P is the sum of the potentials due to the elements of the ring.

Since all points of the ring have the same distance r to P,

$$\Phi_{ring} = -\frac{GmM_{ring}}{r} = -\frac{Gm\sigma(2\pi a\sin\theta)ad\theta}{r}$$

The total potential of the spherical shell at point P can be obtained by summing over all the rings, i.e. integrating the above expression over $\theta \in [0, \pi]$

$$\Phi_{shell} = \int \Phi_{ring} = -\int_0^\pi \frac{2\pi Gm\sigma a^2 \sin\theta d\theta}{r}$$

But from above Fig distance r of P to the ring is

$$r^2 = (r_0 - a\cos\theta)^2 + a^2\sin^2\theta = {r_0}^2 + a^2 - 2ar_0\cos\theta$$

Thus using $M_{shell} = 4\pi a^2\sigma$

$$\Phi_{shell} = -\frac{GmM_{shell}}{2}\int_0^\pi \frac{\sin\theta d\theta}{(r_0{}^2 + a^2 - 2ar_0\cos\theta)^{1/2}}$$

which can be integrated to give

$$\Phi_{shell} = \left[-\frac{GmM_{shell}}{2ar_0}\left(r_0{}^2 + a^2 - 2ar_0\cos\theta\right)^{1/2}\right]_0^\pi$$

Two cases arise:

Point P is outside the shell ($r_0 > a$):

In this case the distance $\left(r_0{}^2 + a^2 - 2ar_0\cos\theta\right)^{1/2}$ which is always +ve is

$$\begin{aligned} (r_0 - a) \quad \text{at} \quad \theta &= 0 \\ (r_0 + a) \quad \text{at} \quad \theta &= \pi \end{aligned}$$

Hence substituting in above integral gives

$$\Phi_{shell} = -\frac{GmM_{shell}}{r_0}$$

That is the potential is the same as that for a point mass M placed at the centre of the spherical shell and similarly the force.

Point P is inside the shell ($r_0 < a$):

In this case the distance

$$\left(r_0{}^2 + a^2 - 2ar_0\cos\theta\right)^{1/2}$$

is

$$\begin{aligned} (a - r_0) \quad \text{at} \;\; \theta &= 0 \\ (a + r_0) \quad \text{at} \;\; \theta &= \pi \end{aligned}$$

Hence substituting in above integral gives

$$\Phi_{shell} = -\frac{GmM_{shell}}{a}$$

Thus the potential inside the sphere is a constant independently of the location of P so the force is zero.

These results can be summarised as:

Newton's sphere theorem: *A spherical shell of uniform density and mass M exerts the same gravitational force on a point mass outside the shell as would a point particle of mass M at the centre. The shell exerts no force on a mass inside it.*

Example: A diametric hole is bored through the centre of the earth (which is assumed to be of uniform density). A stone is released from rest from the surface of the earth. Ignoring the air resistance in the hole describe the resulting motion. Find the time it takes for the particle to travel to the other end of the hole.

[Density of the earth = $5.5 gm/cm^3$ and $G = 6.67 \times 10^{-8} CGS$.]

[Hint: Use Newton's sphere theorem]

9.2 Orbit equation

The above theorem allows the Newton's gravitational force law between point particles to be used in the case of Sun and planets, which are assumed to be spherical bodies with uniform density.

The starting point for deriving the Eq. of orbit of a planet of mass m about the Sun (mass M), a distance r apart, is Newton's 2nd law: $\underline{F} = m\underline{a}$.

We know that orbits under central forces and in particular gravitational field is planar. Thus we use polar coordinates (r, θ) and recall the expressions

$$\begin{aligned} \underline{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e_r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{e_\theta} \\ \underline{F} &= -\frac{GmM}{r^2}\underline{\hat{r}}, \qquad \hat{e_r} \equiv \underline{\hat{r}} \end{aligned}$$

Thus the r & θ components of $\underline{F} = m\underline{a}$ are:

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{GmM}{r^2} \tag{24}$$

$$m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0 \tag{25}$$

Now Eq. (25) can be rewritten as

$$\frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

which implies

$$mr^2\dot{\theta} = J, \quad \text{a const}$$

In fact as we saw before this constant is the magnitude of the angular momentum vector

$$J = |m\underline{r} \times \underline{\dot{r}}| = mr^2\dot{\theta}$$

Eq. (24) can be written more conveniently in terms of a new variable $u = 1/r$. We then have

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

Also

$$\begin{aligned}
\dot{r} &= \frac{dr}{d\theta}\frac{d\theta}{dt} = -r^2\dot{\theta}\frac{du}{d\theta} = -\frac{J}{m}\frac{du}{d\theta} \\
\ddot{r} &= -\frac{J}{m}\frac{d^2u}{d\theta^2}\dot{\theta} = -\frac{J^2}{m^2r^2}\frac{d^2u}{d\theta^2}
\end{aligned}$$

Thus

$$\ddot{r} - r\dot{\theta}^2 = -\frac{J^2}{m^2r^2}\frac{d^2u}{d\theta^2} - \frac{J^2}{m^2r^3}$$

Substituting in (24) we have

$$m\left(-\frac{J^2}{m^2r^2}\frac{d^2u}{d\theta^2}-\frac{J^2}{m^2r^3}\right)=\frac{-GmM}{r^2}$$

which dividing by $-\frac{J^2}{m^2r^2}$ gives

$$\frac{d^2u}{d\theta^2}+u=\frac{Gm^2M}{J^2}\equiv\frac{1}{\ell}$$

where $\ell=\frac{J^2}{Gm^2M}$.

This is the equation of orbit in (u,θ) coords.

It is an inhomogeneous version of SHM equation.

Thus the general solution of this Eq. is

$$u=u_h+u_p$$

where u_h is the general solution of the LHS of this Eq. and u_p is a particular solution of the entire Eq.

From SHM we had that $u_h=A\cos(\theta-\theta_0)$.

It is also easy to check that

$$u_p=\frac{Gm^2M}{J^2}\equiv\frac{1}{\ell}$$

is a particular solution.

Adding this to the solution of the homogeneous part we have **the general solution of the orbit Eq.**

$$u = A\cos(\theta - \theta_0) + \frac{1}{\ell}$$

which can be written as

$$\frac{1}{r} \equiv u = \frac{1}{\ell}\left[e\cos(\theta - \theta_0) + 1\right]$$

where e is an arbitrary constant which needs to be determined from the initial conditions of the orbit.

Above solution represents a conic section.

The following cases arise:

- $e < 1$. This implies $\frac{\ell}{r} > 0$ with r always remaining finite: which represents an **ellipse**

- $e = 1$. This implies $\frac{\ell}{r} \rightarrow 0$ at $(\theta - \theta_0) = \pi$; which represents a **parabola**

- $e > 1$. This implies $\frac{\ell}{r} \rightarrow 0$ at some $(\theta - \theta_0) < \pi$; which represents a **hyperbola**

- $e = 0$. This implies $\frac{\ell}{r} = 1$ or $r = \frac{J^2}{Gm^2M} = const = a$; which represents a **circle**

9.3 Elliptic orbits

We showed that the orbit of planets move on ellipses with the equation ($e < 1$)

$$\frac{1}{r} \equiv u = \frac{1}{\ell}\left[e\cos(\theta - \theta_0) + 1\right]$$

It is useful to write this equation in its more familiar form in Cartesian coords.

Recall the transformations rules between Cartesian to Polar coords:

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Now the elliptic solution is (setting $\theta_0 = 0$ for simplicity)

$$u \equiv \frac{1}{r} = \frac{1}{\ell}\left[e\cos(\theta) + 1\right]$$

which gives

$$\ell = r + re\cos(\theta) = \ell = \sqrt{x^2 + y^2} + xe$$

or

$$x^2 + y^2 = (\ell - ex)^2 = \ell^2 - 2\ell x + e^2 x^2$$

Rearranging

$$x^2(1 - e^2) + 2e\ell x + x^2 = \ell^2$$

or

$$x^2 + \frac{2e\ell x}{1 - e^2} + \frac{y^2}{1 - e^2} = \frac{\ell^2}{1 - e^2}$$

Completing the square by adding $e^2\ell^2/(1 - e^2)^2$ to both sides

$$\left(x + \frac{e\ell x}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{\ell^2}{(1 - e^2)^2}$$

Defining $a = \ell/(1 - e^2)$ and $b^2 = a\ell$ we have

$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is the **equation of an ellipse in Cartesian coordinates** centred at $(-ae, 0)$ with semi-major and semi-minor axes given by a and b.

To see recall

$$
\begin{aligned}
a &= \frac{1}{2}\left[r_{min} + r_{max}\right] = \frac{1}{2}\left[r_{\theta=0} + r_{\theta=\pi}\right] \\
&= \frac{1}{2}\left[\frac{\ell}{1+e} + \frac{\ell}{1-e}\right] = \frac{\ell}{1-e^2}
\end{aligned}
$$

Circular motion: in such motion $|\underline{r}| = r$ is a constant and hence $\dot{r} = 0$. Hence recalling that

$$\underline{v} = \dot{r}\hat{e_r} + r\dot{\theta}\hat{e_\theta}$$

then for a circular motion we have

$$|\underline{v}|^2 = r^2\dot{\theta}^2$$

Orbital period τ

An important observable feature for any closed orbit.

Here wish to connect it to other quantities.

Recall that from Kepler's 2nd law we had

$$\frac{dA}{dt} = \frac{J}{2m} = \text{constant}$$

Integrating over the whole orbit

$$\int_{orbit} dA = \int_0^\tau \frac{J}{2m} dt$$

which gives

$$A = \frac{J}{2m}\tau \qquad \rightarrow \qquad \tau = \frac{2mA}{J}$$

Now for an elliptical orbit the area is

$$A = \pi ab$$

which gives

$$\tau = \frac{2\pi abm}{J}$$

Also for elliptic orbits we have

$$\ell = \frac{J^2}{Gm^2 M}, \quad b^2 = a\ell$$

Squaring and substituting we obtain

$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{a^3}{GM}$$

which shows that

$$\tau^2 \propto a^3$$

i.e. **square of period is proportional to cube of semi-major axis**.

This is Kepler's 3rd law.

This can be used to weigh the Sun! by knowing the semi-major axis of a planet a and its period τ.

Also knowing the ratio of periods of 2 planets we can calculate the ratio of their semi-major axes.

Example: The semi-major axis of orbit of Jupiter is 5.2 times that of Earth. Find its orbital period in years.

Use above law for Earth and Jupiter and divide:

$$\left(\frac{\tau_{earth}}{\tau_{jupiter}}\right)^2 = \left(\frac{a_{earth}}{a_{jupiter}}\right)^3$$

Example: The eccentricity e of an artificial satellite in orbit around the earth is 0.5 and its closest distance to earth is $5R_e$ (where R_e is the radius of the earth). What is its furthest distance from earth, expressed in terms of R_e?

Start with orbit Eq

$$\frac{1}{r} = \frac{1}{\ell}\left[e\cos(\theta - \theta_0) + 1\right]$$

Then r_{min} corresponds to $(\theta - \theta_0) = 0$ which gives

$$\frac{1}{r_{min}} = \frac{1}{\ell}(1 + e)$$

Also r_{max} corresponds to $(\theta - \theta_0) = \pi$ which gives

$$\frac{1}{r_{min}} = \frac{1}{\ell}(1 - e)$$

Therefore

$$\frac{r_{min}}{r_{max}} = \frac{(1 + e)}{(1 - e)} = 3$$

Thus

$$r_{max} = 3r_{min} = 15R_e$$

Example: The orbit of a synchronous communication satellite has been chosen so that viewed from the Earth it appears to be stationary. find the radius of the orbit. $M_e = 5.98 \times 10^{24} kg, \quad G = 6.67 \times 10^{-11} MKS$

Use Kepler's third law

$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{a^3}{GM}$$

and recall that for a synchronous orbit the period of satellite is 24 hours, i.e. $\tau = 86400$ secs.

Substituting for G, M_e and τ we obtain

$$a_{satellite} = 42000 km = 42 \times 10^6 m$$

9.4 Kepler's laws

These laws which were originally obtained on basis of very accurate measurements of Tycho Brahe and others including Kepler. They pre-date Newton's laws and were important for the latter's development and thus of historical interest. Here we have derived them from Newton's laws!

First law: Planets move on elliptic orbits with Sun as a focus

Second law: Equal areas are swept by the line joining the planets and the Sun (i.e. $\underline{r}$) in equal times. (true for any central force)

Third law: The square of the periods of the planets as they orbit the Sun are proportional to the cube of their respective mean distances from the Sun (i.e $\tau^2/a^3 = const$)

Here we have derived them from Newton's laws!