WRG-11 Energy in Wave Motion

Since we would like to consider waves in general, including harmonic waves such as $y(x,t) = A\sin(kx - \omega t)$ which have an infinite extension in x, we shall consider energy per distance λ. That is we are considering the energy in one full oscillation.

For a section of a string between x and $x + dx$,
its mass is approximately μdx

$$\mu \equiv \frac{\text{mass}}{\text{unit length}}$$

Therefore the **kinetic energy** is $dK = \frac{1}{2}\mu dx\left(\frac{\partial y}{\partial t}\right)^2$

and the kinetic energy per unit length is $\frac{dK}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2$. (1)

The **potential energy** is obtained by realising that the tension T acting on the string has extended it from dx to ds.

Hence the potential energy stored in segment dx is $dU = T(ds - dx) = Tdx\left(\frac{ds}{dx} - 1\right)$.

From the above triangle we have $ds^2 = (dx)^2 + (dy)^2 = (dx)^2\left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)$.

Hence $ds = dx\left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)^{\frac{1}{2}} \approx dx\left(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right)$

and so $\left(\frac{ds}{dx} - 1\right) = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2$.

Therefore $\frac{dU}{dx} \approx \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2}\mu v^2\left(\frac{\partial y}{\partial x}\right)^2$ as $v^2 = \frac{T}{\mu}$. (2)

As the wave equation must be obeyed, we can use the general solution $y(x,t) = A\sin(kx - \omega t)$ in equations (1) and (2).

So $\dfrac{\partial y}{\partial t} = -\omega A\cos(kx-\omega t)$ and so $\dfrac{dK}{dx} = \dfrac{1}{2}\mu\omega^2 A^2\cos^2(kx-\omega t)$

$\dfrac{\partial y}{\partial x} = kA\cos(kx-\omega t)$ and so $\dfrac{dU}{dx} = \dfrac{1}{2}\mu k^2 v^2 A^2\cos^2(kx-\omega t)$

But $kv=\omega,$ and so $\dfrac{dK}{dx} = \dfrac{dU}{dx}$.

We see that the kinetic energy density and the potential energy density are equal.

Now we wish to find the energy in one wavelength: $\int_0^\lambda \dfrac{dK}{dx}dx$

$$K_\lambda = U_\lambda = \int_0^\lambda \frac{dK}{dx}dx = \frac{1}{2}\mu\omega^2 A^2\int_0^\lambda \cos^2(kx-\omega t)\,dx \qquad t \text{ is constant.}$$

To calculate the integral, change the variable to $\theta = kx-\omega t$.

Then $d\theta = kdx$.

The limits are now $\begin{array}{ll} x=0, & \theta=-\omega t \\ x=\lambda, & \theta = k\lambda-\omega t = 2\pi-\omega t \end{array}$

So we have

$$\int_0^\lambda \cos^2(kx-\omega t)\,dx = \frac{1}{k}\int_{-\omega t}^{2\pi-\omega t}\cos^2(\theta)\,d\theta = \frac{1}{2k}\int_{-\omega t}^{2\pi-\omega t}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2k}\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_{-\omega t}^{2\pi-\omega t} = \frac{1}{2k}\left[2\pi+\tfrac{1}{2}\left(\sin(4\pi-2\omega t)-\sin(-2\omega t)\right)\right]$$

$$= \frac{\pi}{k}$$

Therefore we have $K_\lambda = U_\lambda = \dfrac{\pi}{2}\dfrac{\mu\omega^2 A^2}{k} = \tfrac{1}{4}\mu\lambda\omega^2 A^2 = \pi^2\mu\lambda f^2 A^2$.

The total energy is $E_\lambda = K_\lambda + U_\lambda = \tfrac{1}{2}\mu\lambda\omega^2 A^2$.

Suppose we sit at a fixed point *x* and measure the total energy per second (power) passing that point, as the wave goes by. Since the wave is travelling at speed *v*, one wavelength λ takes a time $\dfrac{\lambda}{v}$ sec to pass by.

Then power (joules per second) transported in the wave is

$$P = \frac{E_\lambda}{\left(\lambda / v\right)} = \tfrac{1}{2}\mu v \omega^2 A^2$$

Now, using $v = \left(\frac{T}{\mu}\right)^{\frac{1}{2}}$ we define the quantity $\boxed{Z = \mu v = (T\mu)^{\frac{1}{2}}}$ where Z is known as the **Impedance of the String.**

It is entirely a property of the string, while $\omega^2 A^2$ refers to the nature of the propagating wave. The larger Z, the more power is needed to drive the wave of a given frequency and amplitude. Equivalently, for a given power input and frequency, a string of smaller impedance will have a correspondingly larger amplitude. The name comes from electricity where the impedance defines the *resistance* of a circuit for ac voltages.