Wrg-10 Standing Waves

Consider the case of a string fixed at both ends.

This is one of the most famous theoretical problems in the history of Physics:-

1747 D'Alembert, Euler 1755 Daniel Bernoulli
1777 Euler 1807 Fourier

It was required to solve the wave equation $\frac{\partial^2 y}{\partial x^2} - \frac{1}{v_p^2}\frac{\partial^2 y}{\partial t^2} = 0$

for a string, where $v_p = \sqrt{\frac{T}{\mu}}$,

subject to the boundary conditions $y(0,t) = 0$ and $y(L,t) = 0$
at the fixed ends.

Physically we know that each point x on the string vibrates with SHM. Therefore we look for solutions of the form $y(x,t) = A(x)\cos(\omega t + \varphi)$. Note that $y(x,t) = f(x)g(t)$ ……. so *x, t* are not parameters of the same function.

To be a solution of the wave equation we require

$$\left[A''(x) + \left(\frac{\omega}{v_p}\right)^2 A(x)\right]\cos(\omega t + \varphi) = 0, \quad \text{for all t.}$$

Then

$$\boxed{A''(x) = -\left(\frac{\omega}{v_p}\right)^2 A(x) = -k^2 A(x)}$$

which has a solution $A(x) = C\sin(kx + \alpha)$ where *C* is a constant amplitude and α is a constant phase.
Now impose the boundary conditions that the ends of the string are fixed.
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As $y = 0$ at $x = 0$, this implies that $A(0) = 0$. The solution $C = 0$ is not interesting.
Then $\sin(0 + \alpha) = 0$ and so $\alpha = 0$.

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As $y = 0$ at $x = L$, this implies that $A(L) = 0 = \sin(kL)$. Therefore
$kL = n\pi, \quad n = 1, 2, 3, ...$

The only allowed k-values are $k_n = n\dfrac{\pi}{L}, \quad n = 1, 2, 3, ...$

Since $k = \dfrac{\omega}{v_p}$, this limits the angular frequencies to $\omega_n = n\dfrac{\pi}{L}v_p = n\dfrac{\pi}{L}\sqrt{\dfrac{T}{\mu}}$.

These are called the NORMAL MODE FREQUENCIES.

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Let us now picture these normal modes. Firstly note that $k_n = n\dfrac{\pi}{L} = \dfrac{2\pi}{\lambda_n}$.

Therefore $\boxed{\lambda_n = \dfrac{2L}{n}}$ and we plot $y_n(x,0) = C_n \sin\left(n\dfrac{\pi}{L}x\right)$.

$n = 1$ $\qquad$ $\lambda_1 = 2L$

$\lambda_1 / 2$ $\qquad$ L $\qquad$ 2L

$n = 2$ $\qquad$ $\lambda_2 = L$

L

$n = 3$ $\qquad$ $\lambda_3 = \frac{2}{3}L$

L

$n = 4$ $\qquad$ $\lambda_4 = \dfrac{L}{2}$

L

$n = 6$ $\qquad$ $\lambda_6 = \dfrac{L}{3}$

L

We have 6 peaks and 5 nodes, where $y = 0$ for all t.

The n-th normal mode has (n - 1) nodes and n peaks. The time evolution is displayed below. To excite a pure mode, hold the string in the shape above, $\sin\left(n\frac{\pi}{L}x\right)$ and then let go. Plucking will create a complicated shape …. Exciting a linear combination of normal modes.

n = 1 is known as The Fundamental
n = 2 is known as The First Harmonic
n = 3 is known as The Second Harmonic   and so on.

The n-th normal mode has the form:

$y_n(x,0) = C_n \sin\left(n\frac{\pi}{L}x\right)\cos(\omega_n t)$, taking the phase angle as 0 for simplicity.

Note that C_n has different values for each n-value. As n becomes larger C_n becomes smaller and contributes less to the general waveform. So the number of terms needed is limited.

The general solution of the wave equation is the Fourier Series

$$y_n(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(n\frac{\pi}{L}x\right)\cos(\omega_n t)$$

To show that standing wave on a string are running waves (travelling waves).

The pure normal modes appear to *stand still* and just go up and down. The nodes remain in the same place. Hence the name standing waves. { In Quantum Mechanics, states of definite energy are the *normal modes* and are known as stationary states.}

n = 1

n = 2

n = 3

n = 5

Vibrations of a string in various simple modes (n = 1, 2, 3, 5)

In contrast, linear superpositions of several normal modes seemed to move back-and-forth along the string (as well as up-and-down). In fact, **in all cases**, the vibrations of a string are made up of superpositions of running waves and the vibrations seen are the result of constructive and destructive interference between the running waves.

What happens to a running wave at the ends of the string?
They are reflected, undergoing a sign reversal in amplitude, and change of direction. This is because the fixed ends exert an equal and opposite force on the string to the one the string exerts on the fixing. Clearly these forces have to cancel exactly, giving complete destructive interference at the ends.

We are now in a position to interpret standing waves on a string in a physically useful manner as the superposition of equal, but opposite signed amplitude running waves, propagating in the opposite directions.

Consider a pure normal mode,

$$y_n(x,0) = C_n \sin\left(n\frac{\pi}{L}x\right)\cos(\omega_n t) = C_n \sin(k_n x)\cos(\omega_n t) \text{ where } k_n = n\frac{\pi}{L},\ \omega_n = v_p k_n .$$

But

$$\sin A \cos B = \tfrac{1}{2}\{\sin(A-B) - \sin(A+B)\}$$

Therefore $$y_n(x,t) = \frac{C_n}{2}\{\sin(k_n x - \omega_n t) - \sin(k_n x + \omega_n t)\}.$$

The first term is right-moving wave with an amplitude of C_n and the second term is a left-moving wave with the same magnitude of amplitude, but of opposite sign.

To reiterate, we see that the standing wave pattern on a string in a normal mode results from the superposition of left and right moving equal, but opposite signed, amplitude waves. These in turn result from waves being repeatedly and perfectly reflected from the rigidly fixed ends of the string at $x = 0$, and $x = L$. This is the mechanism that sustains the two counter-moving running waves.

The standing wave is the superposition of these two, as illustrated below.
Note how the waves pass through each other, interfering to produce for the human eye the appearance of standing waves.
The nodes arise from the permanent destructive interference, at these particular positions.

Two waves travelling in opposite directions and their resultant standing wave, with time.

v

sum

v

sum

v

sum

The time evolution of a standing wave is shown below. Note that, at first sight, this introduces a puzzle. Since the vibrations of the string represent both kinetic and potential energy, where has all the energy gone at the $t = 2T/8$?

The answer is that all the energy is kinetic …. All points on the string are moving at their highest velocity, with each point vibrating about $y = 0$ with SHM.

Standing waves in air columns

We cannot have a taut string with one end open, but for air columns we can. Vibrations in air are longitudinal pressure waves, which we will introduce later. For convenience the displacement of the air, although in the x -direction, is plotted symbolically in the y-direction.
For standing waves solutions, the closed end must be a place of zero displacement. The open end however must be a place of maximum displacement.

Clearly the possible standing waves are as shown below. For the n-th normal mode, closed one end and open the other, we have $k_n = \frac{\pi}{L}\left(n - \frac{1}{2}\right)$. Similar formulae can be obtained for the case of open both ends, and closed both ends.

First three normal modes with tube open at one end only.

First three normal modes with tube open at both ends.

Example for n = 2, with one end open.

$$\frac{3}{4}\lambda = L \quad \text{and} \quad \lambda k = 2\pi$$

So $\frac{3}{4}\lambda k = \frac{3\pi}{2} = kL$. Hence $k = \frac{3\pi}{2L}$, or $k_n = \frac{\pi}{L}(n - \frac{1}{2})$, where $n = 2$.