**Wrg-09** **Solving the Wave Equation.**

We have already noticed that beads on a string move with SHM (the Normal Modes), or with the superpositions of SHM (superposition of Normal Modes).

We also noted that in the limit of infinitely many beads, we obtain the solution for a continuous string. This suggests that if a continuous string vibrates in a normal mode, each point vibrates with SHM at a definite frequency $\omega$, the normal mode frequency.
Therefore we search for a solution to the wave equation with the real form $\cos\omega t$.

It simplifies the search to use the complex form:

$$y(x,t) = A(x)e^{i\omega t}$$

$A$ is the amplitude at position $x$ on the string, and the exponential term represents the harmonic vibration of point x on the string.

Differentiating

by $t$ $$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A(x)e^{i\omega t}$$

by $x$ $$\frac{\partial^2 y}{\partial x^2} = \frac{d^2 A(x)}{dx^2} e^{i\omega t} \equiv A''(x)e^{i\omega t}$$

To be a solution to the wave equation we require that:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$$

Therefore $$\left[A''(x) + \left(\frac{\omega}{v}\right)^2 A(x)\right]e^{i\omega t} = 0, \quad \text{for all t.}$$

$$\boxed{A''(x) = -\left(\frac{\omega}{v}\right)^2 A(x) = -k^2 A(x)}$$

where we define the **wave number** $k$ by $k^2 = \left(\frac{\omega}{v}\right)^2$ or $k = \pm\frac{\omega}{v}$

Note that units of $k$ are $[\mathrm{m}]^{-1}$.

Mathematically this looks like a solution of SHM and so we can write
$A(x) = Ae^{+ikx}$ and $Be^{-ikx}$
So the solution is of the form $Ae^{+i(kx+\omega t)}$ and $Be^{-i(kx-\omega t)}$

Hence possible forms are $y(x,t) = \sin(kx \pm \omega t)$ and $\cos(kx \pm \omega t)$ or any linear combinations.

Notice that the wave equation is linear in $x$ and homogeneous and so the Superposition Principle applies to the solutions.

A more general solution is
$y(x,t) = A\sin(kx+\omega t) + B\ \cos(kx+\omega t) + C\sin(kx-\omega t) + D\ \cos(kx-\omega t)$ but even this is not the most general solution.

**The most general solution**
It is actually $y(x,t) = f(kx-\omega t) + g(kx+\omega t)$ where *f* and *g* are any functions whatever.

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Let us prove that statement:

$$\frac{\partial f}{\partial t} = \frac{\partial(kx-\omega t)}{\partial t}\frac{df(kx-\omega t)}{d(kx-\omega t)}$$ using the chain rule.

$\frac{\partial f}{\partial t} = -\omega f'$ Note that $f'$ means differentiate with respect to the argument of $f$ which, here, is $kx-\omega t$.

Similarly $\frac{\partial^2 f}{\partial t^2} = \omega^2 f''$ and $\frac{\partial^2 g}{\partial t^2} = \omega^2 g''$

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Also $$\frac{\partial f}{\partial x} = \frac{\partial(kx-\omega t)}{\partial x}\frac{df(kx-\omega t)}{d(kx-\omega t)}$$ using the chain rule.

$\frac{\partial f}{\partial x} = k f'$ Note that $f'$ means differentiate with respect to the argument of $f$ which, here, is $kx-\omega t$.

Similarly $\dfrac{\partial^2 f}{\partial x^2} = k^2 f''$ and $\dfrac{\partial^2 g}{\partial x^2} = k^2 g''$

To be a solution of the wave equation we require

$$\left[k^2 - \left(\frac{\omega}{v}\right)^2\right] f'' + \left[k^2 - \left(\frac{\omega}{v}\right)^2\right] g'' = 0$$

But since $k^2 = \left(\dfrac{\omega}{v}\right)^2$, each term equals zero independently of *f* and *g*.

The harmonic solutions of the form

$f(kx - \omega t) = A\sin(kx - \omega t)$ and $g(kx + \omega t) = B\sin(kx + \omega t)$

are just special cases …… and cosines too.

Fourier's theorem tells us that any functions *f* and *g* can be expressed as a linear combination of sines and cosines, giving an infinite number of terms, each with a different frequency. Only the early terms in the series have large amplitude and are therefore significant. This theorem is a sophisticated version of the Superposition Principle and is the reason that we need only confine our considerations to a single frequency.

$\sin(kx \pm \omega t)$ is a typical term in the full Fourier Series.

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## Properties of the travelling wave

Consider the harmonic wave $y(x,t) = A\sin(kx - \omega t)$

amplitude phase

At $t = 0$, the profile of the wave is

$kx = 0 \quad kx' = \frac{\pi}{2}$ (value of phase at this position at $t = 0$.)

The wavelength $\lambda$ is the repeat distance ….. that is the distance over which the phase changes by $2\pi$.

$$\Delta(\text{phase}) = 2\pi = [k(x' + \lambda) - \omega t] - [kx' - \omega t]$$

At the same time $t$ ……… phase at $x' + \lambda$ and phase at $x'$

So $\Delta(\text{phase}) = k\lambda$ and so $\boxed{k\lambda = 2\pi}$

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**The phase velocity** $v_p \equiv$ **velocity of points of constant phase.**

Point of constant phase is defined by,

$$\text{phase} \equiv kx - \omega t = \text{ constant.}$$

Differentiate with respect to $t$,

$$k\frac{dx}{dt} - \omega = 0$$

Therefore $v_p \equiv \dfrac{dx}{dt} = \dfrac{\omega}{k}$ or $k = \dfrac{\omega}{v_p}$

But $v = \left(\dfrac{T}{\mu}\right)^{\frac{1}{2}}$ for a string is also given as $k = \dfrac{\omega}{v}$, therefore $v = v_p$, the phase velocity.

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Or again consider the wave profile at time $t$ and at a later time, $t$+ d$t$.

d$x$ is the distance travelled by the peak in time d$t$.

The phase here is $\dfrac{\pi}{2} = k(x + dx) - \omega(t + dt)$

Subtracting we obtain

$$\frac{\pi}{2} - \frac{\pi}{2} = 0 = kdx - \omega dt$$

$$dx = \frac{\omega}{k}dt$$

Therefore the phase velocity of the wave is $v_p = +\frac{\omega}{k}$,
and we note that this is a *right* moving wave

$$y(x,t) = A\sin(kx - \omega t)$$

To show that $y(x,t) = A\sin(kx + \omega t)$ is a left-moving wave.

The point of constant phase is one for which $kx + \omega t = \text{constant}$.

Then $x = -\frac{\omega}{k}t + \frac{\text{constant}}{k}$

Therefore $v_p \equiv \frac{dx}{dt} = -\frac{\omega}{k}$

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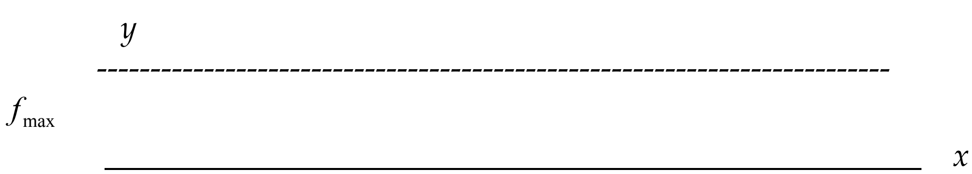
The same arguments apply to any wave *obeying the wave equation*, such as $f(kx - \omega t)$ or $g(kx + \omega t)$

Suppose at t = 0, this is the picture of $y(x, t = 0) = f(kx)$ which a maximum at $x = x_0$, $f_{\max} = f(kx_0)$.

Now let the wave pulse travel for a time $t_1$ to $x_1$.




This is a picture of $f(kx_1 - \omega t_1)$ which has its peak at $x_1$ where

$$f(kx_1 - \omega t_1) = f_{\max} = f\left(kx_0\right)$$
$$f(kx_1 - \omega t_1) = f\left(kx_0\right)$$
$$kx_1 - \omega t_1 = kx_0 \quad \text{or} \quad \frac{(\mathrm{x}_1 - x_0)}{t_1} = +\frac{\omega}{k} = v_p$$