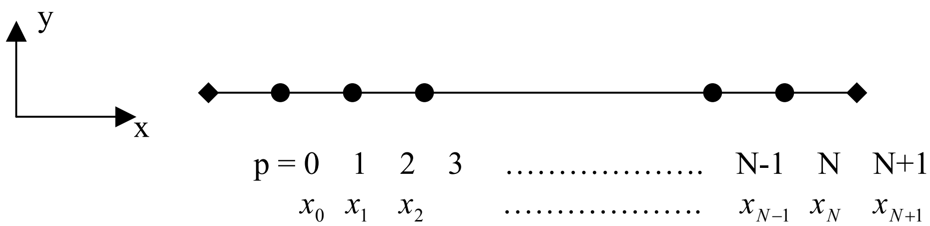
Wrg-08 Many Coupled Oscillators …. The Wave Equation

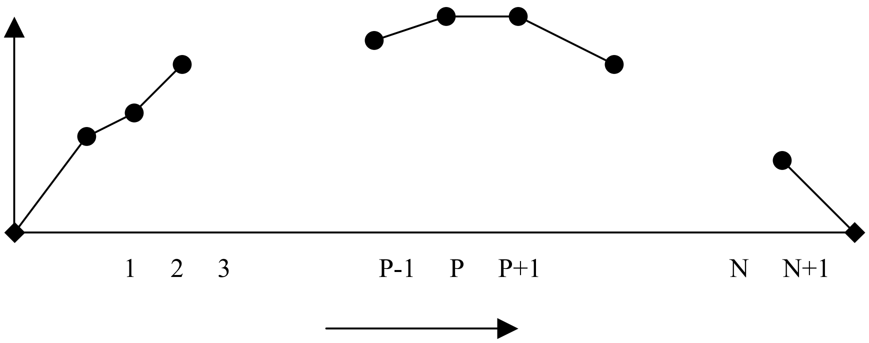
Consider a flexible massless string, with N beads of equal mass, spaced out with a bead separation ℓ, and ends (p = 0, N+1) fixed.

With zero displacements in y we have this diagram of the situation.

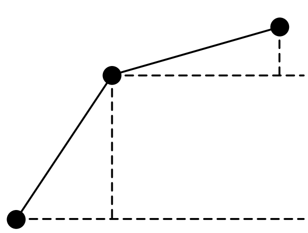


We consider transverse (y-direction) oscillations so small that the tension, T, in the string is approximately constant throughout. Also the length of the string remains approximately constant. The angles α_p are so small that $\sin\alpha_p \approx \alpha_p$ in radians and $\cos\alpha_p \approx 1$. At an arbitrary time t, the diagram below shows the string, but with the displacements exaggerated.

Note that $y_0 = 0$ at $x = x_0 = 0$ and $y_{N+1} = 0$ at $x = x_{N+1}$.



Consider p^{th} mass: F_{y_p} = upwards force on pth mass $= -T\sin\alpha_{p-1} + T\sin\alpha_p$



$$\sin\alpha_{p-1} \approx \frac{y_p - y_{p-1}}{l} \qquad (1a)$$

$$\sin\alpha_p \approx \frac{y_{p+1} - y_p}{l} \qquad (1b)$$

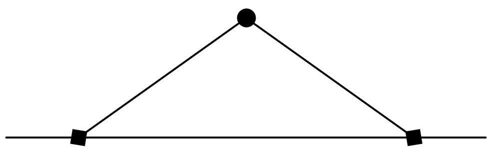
Using Newton's second Law for p^{th} mass: $p = 1, 2, 3, \ldots\ldots N$.

$$m\ddot{y}_p = F_{yp} = \frac{T}{l}\left(y_{p+1} - y_p\right) - \frac{T}{l}\left(y_p - y_{p-1}\right)$$
$$\text{or,} \quad \ddot{y}_p = +\omega_0^2\left(y_{p+1} - 2y_p + y_{p-1}\right) \quad \text{where} \quad \omega_0^2 = \frac{T}{ml} \tag{2}$$

Let's consider special cases one-by-one:

N=1

Equation 2. is only one equation with P = 1, and $y_0 = y_2 = 0$.

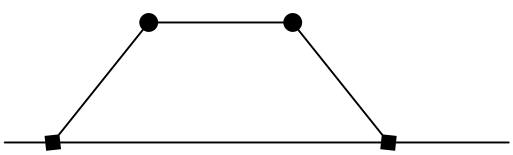


$$\ddot{y}_1 = -2\omega_0^2 y_1$$

The particle oscillates up and down transversely with $\omega_1 = \sqrt{2}\omega_0 = \sqrt{\frac{2T}{m\ell}}$

N=2

Again, with $y_0 = y_3 = 0$, equation 2. is two equations (P = 1,2)



$$\ddot{y}_1 = \omega_0^2\left(y_0 - 2y_1 + y_2\right) = -2\omega_0^2 y_1 + \omega_0^2 y_2$$
$$\ddot{y}_2 = \omega_0^2\left(y_1 - 2y_2 + y_3\right) = \omega_0^2 y_1 - 2\omega_0^2 y_2$$

Let's solve these for normal frequencies and normal nodes. Assume the same ω for each bead.

Letting $y_1 = A_1 e^{i\omega t}$, $y_2 = A_2 e^{i\omega t}$ gives

$$\left(-\omega^2 + 2\omega_0^2\right)A_1 = \omega_0^2 A_2$$
$$\left(-\omega^2 + 2\omega_0^2\right)A_2 = \omega_0^2 A_1$$

Dividing to eliminate the A_1 and A_2:
$$\frac{\left(-\omega^2 + 2\omega_0^2\right)}{\omega_0^2} = \frac{\omega_0^2}{\left(-\omega^2 + 2\omega_0^2\right)}$$

$$\left(-\omega^2 + 2\omega_0^2\right)^2 = \omega_0^4 \qquad \left(-\omega^2 + 2\omega_0^2\right) = \pm\omega_0^2$$

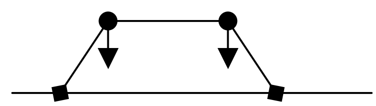
so $\omega^2 = \omega_0^2$ or $3\omega_0^2$

Therefore $\omega = \omega_0$ and $\omega = \sqrt{3}\omega_0$ are the normal frequencies.

For $\omega = \omega_0$, $A_1 = \mathrm{A}_2$ and for $\omega = \sqrt{3}\omega_0$, $A_1 = -A_2$

Consider $\omega = \omega_0$, $A_1 = A_2$.
Taking the phases = 0, ie $y_1 = y_2 = 0$ at $t = 0$, we have for the normal modes:

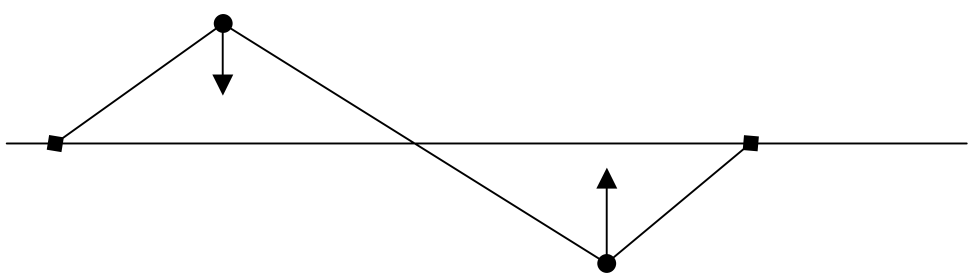
$y_1 = A_1 \cos\omega_0 t$ and $y_2 = A_1 \cos\omega_0 t$, so $y_1 = y_2$



The masses oscillate in phase with the frequency ω_0 between the limits shown in the diagram.

Consider $\omega = \sqrt{3}\omega_0$, A_1 = -A_2 .

$y_1 = A_1 \cos\sqrt{3}\omega_0 t$ and $y_2 = -A_1 \cos\sqrt{3}\omega_0 t$, so $y_1 = -y_2$



The masses oscillate precisely out of phase and, as you might expect, with the higher frequency.

N=3 We will not consider here the further cases, but we can see how the possibilities develop as N increases. The highest frequency mode will always be where alternate masses have displacements on opposite sides of y = 0; The lowest frequency mode is where they are all moving in phase, producing the smoothest possible curve when joined together.

N=5

Note that $L = (N+1)\ell$ and $x_p = p\ell = x$. n is the mode.

For n = 1, beads lie on curve $\sin\left(\frac{x_p}{L}\pi\right) = \sin\left(\frac{p\ell}{(N+1)\ell}\pi\right) = \sin\left(\frac{p\pi}{N+1}\right)$

For the n[th] mode, we have that the curve is $\sin\left(\frac{n\pi}{L}x\right)$.

modes

n = 1 ______________________________

n = 2 ______________________________

The string joining the beads are straight lines

n = 3 ______________________________

n = 4 ______________________________

n = 5 ______________________________

From the foregoing we can see that as the number N of masses gets large and their separation ℓ gets small, the distribution of points in the normal modes of oscillation starts to look almost continuous. Thus, if we let $N \to \infty$ and $m, \ell \to 0$ we can think of the beads as the atoms in a string. We also note that in this limit:

$$\frac{y_{p+1} - y_p}{l} \underset{l=0}{\longrightarrow} \left(\frac{dy}{dx}\right)_{x=x_{p+1}} \quad \text{and} \quad \frac{y_p - y_{p-1}}{l} \underset{l=0}{\longrightarrow} \left(\frac{dy}{dx}\right)_{x=x_p} \quad \text{where} \quad x_{p+1} - x_p = l\cos\alpha_p \approx l.$$

and, in eq 2 for p,

$$\frac{d^2 y_p}{dt^2} = \frac{T}{m}\left[\left(\frac{dy}{dx}\right)_{x=x_{p+1}} - \left(\frac{dy}{dx}\right)_{x=x_p}\right]$$

$$\frac{d^2 y_p}{dt^2} = \frac{T}{\left(m/l\right)}\left[\left(\frac{dy}{dx}\right)_{x=x_{p+1}} - \left(\frac{dy}{dx}\right)_{x=x_p}\right] \div l$$

The difference of two derivatives at neighbouring points, divided by ℓ, becomes the 2nd derivative as $l \to 0$.

Therefore $$\left(\frac{d^2y}{dt^2}\right)_{x=x_p} \underset{l\to 0}{\longrightarrow} \frac{T}{\mu}\left(\frac{d^2y}{dx^2}\right)_{x=x_p}$$

where $\mu = \dfrac{m}{l}$ is the mass per unit length of our string, and is finite.

Since y clearly depends on both time, (t) and position along the string (x) we should use partial derivatives, giving

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y(x,t)}{\partial x^2} \quad \text{or} \quad \boxed{\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0} \quad \text{where} \quad v = \sqrt{\frac{T}{\mu}} \text{ is a velocity.}$$

This is the **WAVE EQUATION**, in this case for a string having tension T and mass/unit length μ.

Note: (1) $[\mu] = kg.m^{-1}$ $\qquad [T] =$ [Newtons] $= kg.m.s^{-2}$

Therefore $$\left[\frac{T}{\mu}\right] = \frac{kg.m.s^{-2}}{kg.m^{-1}} = m^2s^{-2} = \left[v^2\right]$$

Note: (2) We notice that the transition $N \to \infty, \quad l \to 0$ corresponding to a continuous distribution of mass on the string, means that there is an infinity of normal modes of vibration. Thus, the most general vibration of the string is an infinite sum of normal modes:

$$y(x,t) = \sum_{n=1}^{\infty} a_n y_n(x,t)$$

where $y_n(x,t)$ is the n$^{\text{th}}$ mode with $\omega = \omega_n$.

In fact, $y_n(x,t) = \sin\left(\dfrac{n\pi}{L}x\right)\cos(\omega_n t)$ where L is total length of string,

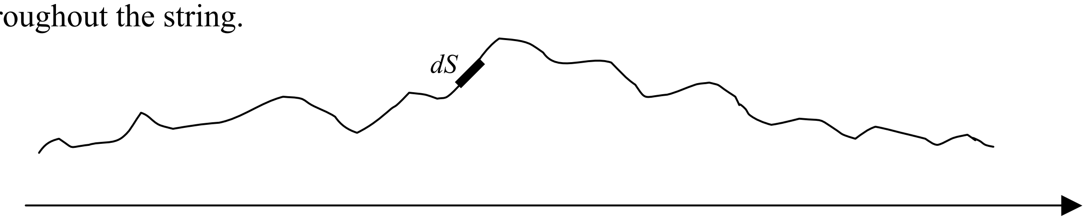
and $\omega_n = \dfrac{n\pi}{L}\left(\dfrac{T}{\mu}\right)^{\frac{1}{2}} = n\omega_1$ where ω_1 is the lowest frequency (the fundamental), as we shall see later.

This is an instance of Fourier's Theorem, that any periodic function (such as *y(x,t)*) can be expanded in terms of sines and cosines with frequencies in integral multiples of that of *y(x,t)*.
[This applies to periodicity in both time (t) and space (x).]

We have now seen that the particles (atoms) in a vibrating string oscillate up and down. Only in the normal modes is this SHM. Otherwise it is a linear superposition of normal mode SHMs, having the form $y_n(x,t) = \sin\left(\frac{n\pi}{L}x\right)\cos(\omega_n t)$. We now show this directly by studying the continuous string:-

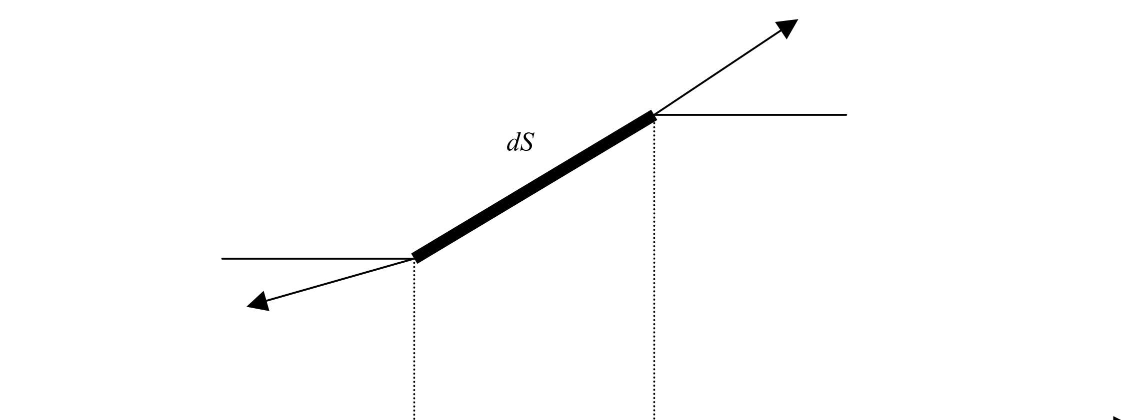
The Wave Equation for a Continuous String

The limit $N \to \infty$ of N beads on a string corresponds to a continuous string with uniform mass/unit length, μ.
In the first instance we consider a small portion of an infinitely long string, displaced from its zero displacement position by a very small amount. There is no motion of the string in the x-direction, and therefore $F_x = 0$. This makes the tension T constant throughout the string.



The segment experiences a net downward restoring force $F_{y,}$ but no horizontal force, $F_x \approx 0$.

Segment of a string under tension *T*.

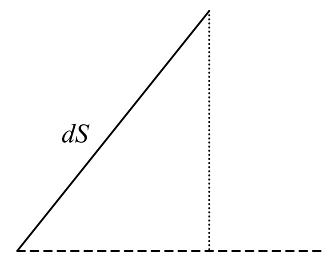


$F_x = T\cos(\theta + d\theta) - T\cos\theta \approx T - T = 0$
$F_y = T\sin(\theta + d\theta) - T\sin\theta \approx T(\theta + d\theta) - T\theta$
So $F_y \approx Td\theta$

Now express $d\theta$ in terms of dx (In fact $d\theta$ depends on the curvature of the string, $\frac{\partial^2 y}{\partial x^2}$).

From diagram we see that $\tan\theta = \frac{\partial y}{\partial x}$.

Differentiating we get $$\frac{d(\tan\theta)}{dx} = \frac{d(\tan\theta)}{d\theta}\frac{d\theta}{dx} = \sec^2\theta\frac{d\theta}{dx} = \frac{\partial^2 y}{\partial x^2}$$



But $\sec\theta = \frac{1}{\cos\theta} \approx 1$ for small θ.

Therefore $$\boxed{d\theta \approx \frac{\partial^2 y}{\partial x^2}dx}$$

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We also require the element of mass *dm* in terms of the length element.

For a segment of length, $ds = \frac{dx}{\cos\theta} \approx dx$, and $dm \equiv$ segment mass $= \mu ds$.

Therefore $$\boxed{dm \approx \mu dx}$$

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So for a segment of string of mass *dm*, at position *x* and time *t*,

$$F_y \approx Td\theta = T\frac{\partial^2 y}{\partial x^2}dx = dm\frac{\partial^2 y}{\partial t^2} = \mu\frac{\partial^2 y}{\partial t^2}dx$$

The *dx* cancel, giving the **Wave Equation**, for a uniform continuous string.

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0 \text{ where, as before, } \quad v = \sqrt{\frac{T}{\mu}} .$$

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What is it that v is a speed of? As we will see later it the phase velocity of the wave.

The wave equation occurs throughout Physics, with v depending on different quantities for each individual case. The wave velocity definition changes depending on the type of wave.

Sound waves in a gas $\quad v = \left(\frac{\gamma P}{\rho}\right)^{\frac{1}{2}} \approx 336 m.s^{-1}$ (air at STP)

Vibrational waves in a solid $\quad v = \left(\frac{Y}{\rho}\right)^{\frac{1}{2}} \approx 4714 m.s^{-1}$ (Al rod)

Electromagnetic waves $\quad v = \frac{c}{n} \approx 3 \times 10^{8}\ \mathrm{m.s^{-1}}$ (in vacuum) n = refractive index

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