Wrg-07 Two Coupled Oscillators

Now that we have discussed one way in which an oscillator should be made more realistic - damping and driving - we now go on to another aspect of the same attempt towards realism. Rarely do we find oscillators in nature completely isolated; they are usually *coupled* in some way to their surroundings (via non-rigid supports) or to their neighbours, all of which are themselves oscillators.

Practical examples are everywhere in nature, in the laboratory and in technology:
optical table attached to a vibrating floor; different components of a space ship superstructure; components of musical instruments; more esoteric examples such as $K^0\overline{K}^0$ meson oscillations (see 3rd year Elementary Particle Physics course) and neutrino oscillations in the sun and other astronomical situations; oscillations of atoms in diatomic, tri-atomic and more complicated molecules; vibrations of atoms in a solid where, in both these latter cases, the coupling occurs through the interatomic forces; coupled circuits, where coupling can be via mutual inductance or capacitive coupling; many atomic and molecular systems which, when isolated, have identical energy levels but in interaction these split - this is the degeneracy to non degeneracy transition: two famous examples are the ***COVALENT BOND*** in molecules and the creation of ***ENERGY BANDS*** in solids.

I have placed this topic at this point in the course not only as a further example of making our considerations more realistic, but also because it leads naturally to a consideration of *very many* coupled oscillators: this is how ***RUNNING WAVES*** propagate in a medium such as a solid, gas or liquid. Thus we will be led to a natural understanding of how *energy* can propagate through a material medium without the need to transport the medium itself: energy gets transferred from atom to atom via their coupling.

DEMONSTRATION: We begin with a demonstration of a) two vertically oscillating slinkys with a light chain joining them producing a weak coupling and b) two pendulums with a coupling between them.

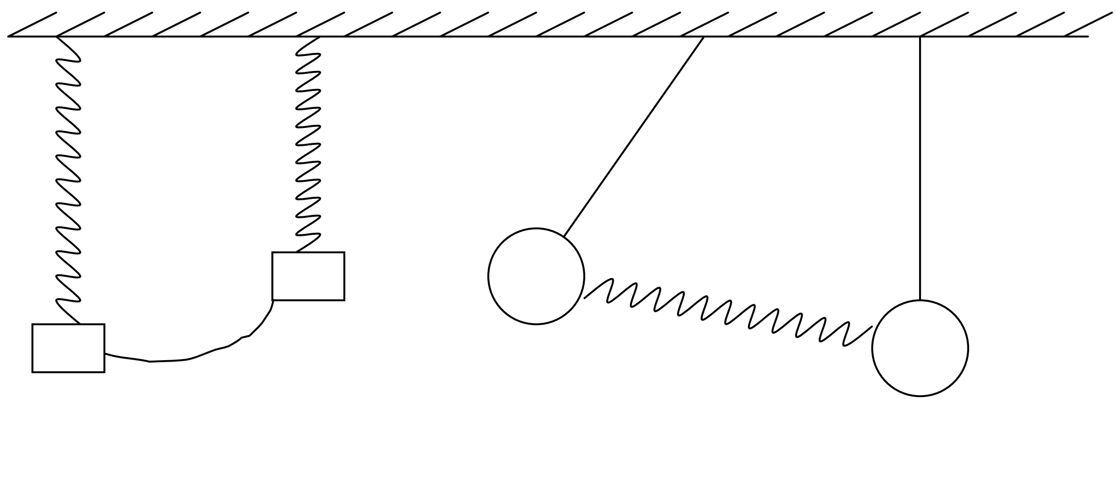

Figure 7.1: Weakly coupled oscillators.

We initially set A oscillating, with B initially undisturbed. Almost immediately the coupling causes B to commence oscillating, at first only with a small amplitude. But as time passes we find that B oscillates with ever increasing amplitude with A's decreasing exactly in step.

Eventually B is seen to absorb *all* the energy of A which, for a brief instant, appears to stop oscillating. Thereafter B's energy gets transferred back to A. B again comes to rest for an instant, until the system has reproduced its initial configuration before going through the entire sequence again. This back-and-forth energy transfer continues in this periodic manner until the damping in the realistic system eventually damps out the oscillations. In an undamped system this back-and-forth transfer of energy persists undiminished.

Our aim is to understand this behaviour. In the process we will uncover a richness and depth quite unexpected in such an apparently simple dynamical system. As always we will begin our study with a simple idealised (linear and zero damping) model: horizontal oscillations of two equal masses sliding on a frictionless table connected by a symmetrical arrangement of springs:

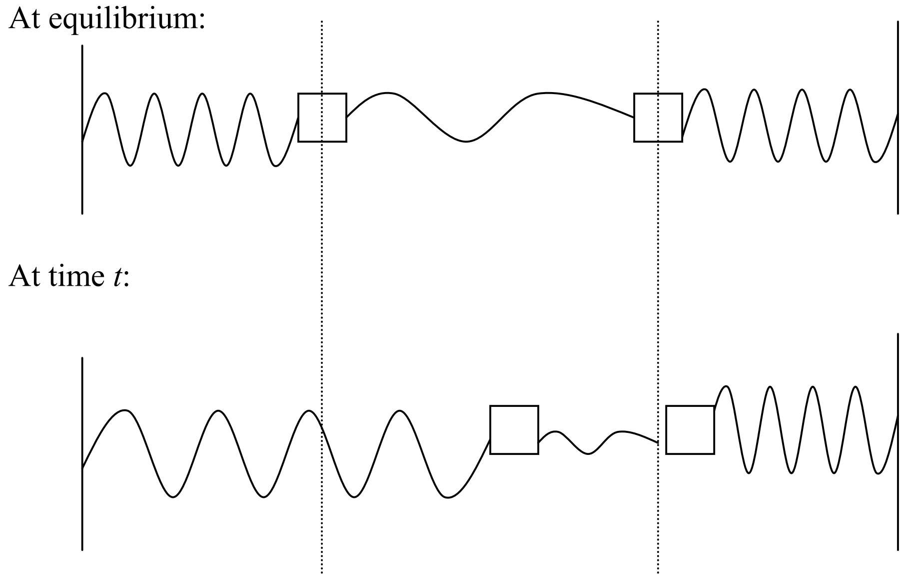

Figure 7.2 Two coupled horizontal oscillators.

F_x	$=$	$-s_0 x$	$+$	$s(-x+y)$
Force on left mass in positive x-direction		left spring stretched by x		coupling spring compressed by x and stretched by y

F_y	$=$	$-s_0 y$	$-$	$s(-x+y)$
Force on right mass in positive y-direction		right spring compressed by y		coupling spring compressed by x and stretched by y

Notice that the coupling spring exerts equal *and opposite* forces on the two masses, as required by the law of equal and opposite action and reaction.

The equations of motion (Newton II) for each of the two masses are then:

For the left mass $\quad m\ddot{x} = F_x = -s_0 x + s(-x + y) \qquad (1)$

For the right mass $\quad m\ddot{y} = F_y = -s_0 y - s(-x + y) \qquad (2)$

Notice that the variables *x* and *y* are mixed up in each other's equation:

Equation (1) is a differential eq. for *x* but also contains the other variable *y*; similarly, equation (2) for *y* contains *x* also. This, of course, is a consequence of the coupling. To solve we need to somehow achieve a differential equation involving only one variable. These variables will be certain linear combinations of *x* and *y* known as the ***NORMAL COORDINATES***, but in general it's not easy to find these by guesswork or by inspection. In this particular case with its high degree of symmetry (equal masses, with identical s_0 springs) it's obvious how to proceed: if we add the equations we find a neat cancellation of the (*x-y*) term and the result is an equation for the quantity (*x+y)* alone.

Similarly, by subtracting the equations we find an equation for (*x-y*) alone. For this system these are the ***NORMAL COORDINATES*** we are seeking:

$$X \equiv x + y \qquad (3)$$

$$Y \equiv x - y \qquad (4)$$

and they obey simple harmonic oscillator equations:

From eq.(1) + eq.(2):

$$m(\ddot{x} + \ddot{y}) = -s_0(x + y)$$

$$\ddot{X} = -\left(\frac{s_0}{m}\right)X$$

$$\ddot{X} = -\omega_0^2 X \quad \text{where} \quad \omega_0 = \sqrt{\frac{s_0}{m}} \qquad (5)$$

From eq.(1) - eq.(2):

$$m(\ddot{x} - \ddot{y}) = -s_0 x + s_0 y - sx + sy - sx + sy$$

$$= -(s_0 + 2s)(x - y)$$

$$\ddot{Y} = -\left(\frac{s_0 + 2s}{m}\right)Y$$

$$\ddot{Y} = -\omega_1{}^2 Y \quad \text{where} \quad \omega_1 = \sqrt{\frac{(s_0 + 2s)}{m}} \qquad (6)$$

Note that $\omega_1 > \omega_0$.

These equations, being SHM equations for the NORMAL COORDINATES X and Y, with respective ***NORMAL MODE FREQUENCIES*** ω_0 and ω_1 have general solutions,

$$X(t) = A_0 \cos(\omega_0 t + \varphi_0) \qquad (7)$$
$$Y(t) = A_1 \cos(\omega_1 t + \varphi_1) \qquad (8)$$

These are also called ***EIGENMODES*** and the normal mode frequencies**,** ***EIGENFREQUENCIES***. In Quantum Mechanics these frequencies are related to energies via $E = hf = \hbar\omega$, and correspond to the quantised energies or ***ENERGY EIGENVALUES*** that appear in atoms, molecules, nuclei, solids and even in elementary particle physics.

By adding and subtracting and dividing by 2 we then obtain the general solution to our original problem:

$$x(t) = \tfrac{1}{2}(X+Y) = \frac{A_0}{2}\cos(\omega_0 t + \varphi_0) + \frac{A_1}{2}\cos(\omega_1 t + \varphi_1) \qquad (9)$$
$$y(t) = \tfrac{1}{2}(X-Y) = \frac{A_0}{2}\cos(\omega_0 t + \varphi_0) - \frac{A_1}{2}\cos(\omega_1 t + \varphi_1) \qquad (10)$$

The lesson we draw from this result is a far-reaching one*: the most general form of vibration is a superposition of the NORMAL MODES of vibration of the system*.

This enormous simplification generalises to all types of complicated systems including those with more than two coupled particles and even with infinitely many:
with N coupled oscillators we obtain N normal modes. Thus with 10^{23} atoms coupled in a solid crystal the binding forces act as coupling springs and we obtain 10^{23} normal modes which can be investigated experimentally by infrared spectroscopy and determine the specific heat in Einstein's and Debye's quantum theories of specific heats.

In a *continuous* system such as a vibrating string or a fluid where the discontinuity of atomic structure can usually be ignored there are effectively infinitely many oscillators joined together, so $N \to \infty$, and we have infinitely many normal modes. These are the terms in a Fourier series or Fourier integral which enables us to study such complicated systems as musical instruments, concert halls, to name only a few examples.

PHYSICAL SIGNIFICANCE: How to excite the normal modes.

The physical significance of these normal modes is buried in eqs (7) to (10): the sum X oscillates with frequency ω_0; the difference Y oscillates with frequency $\omega_1 \rangle \omega_0$, and clearly we need to choose either A_0 or A_1 to be zero to excite only one mode. What we need to do is to choose just the right *initial conditions* (at $t = 0$) to excite only one of the normal modes of vibration:

(1) **Exciting the lowest mode**, ω_0, with the normal coordinate X : Since the normal co-ordinate for the other mode is Y, we have to choose initial conditions to ensure that $Y(t) = 0$ at all times. This is accomplished by choosing

$$Y(0)=0 \quad \text{and using} \quad Y(t)=A_1\cos(\omega_1 t+\varphi_1) \qquad Y(0)=0=A_1\cos\varphi_1$$
$$\dot{Y}(0)=0 \quad \text{and using} \quad \dot{Y}(t)=-\omega_1 A_1\sin(\omega_1 t+\varphi_1) \qquad \dot{Y}(0)=0=-\omega_1 A_1\sin\varphi_1 \qquad (11)$$

What do these conditions imply?
The first equation is solved by either $A_1=0$ or $\varphi_1=\pi/2$; the second by $A_1=0$ or $\varphi_1=0$. Clearly the only possibility is therefore $A_1=0$, which ensures $Y(t)=0$ for all *t*.

Since *Y=(x-y)*/2, the way to ensure that *Y(t)*=0 for all times is to ensure that *x(t)=y(t)* for all times, including *t* = 0 of course. The way to accomplish this is to set up initial conditions with $x(0)=y(0)$ and $\dot{x}(0)=\dot{y}(0)$; this then ensures *Y*(0)=0 at all subsequent times, ie. *Y*(*t*)=0.

Then, to excite the ω_0 mode all we need do is excite vibrations in *X* by choosing an initial displacement $X(0)=2a$ and, for simplicity, an initial velocity $\dot{X}(0)=0$:

$$\left\{\begin{array}{lll} X(0)=2a & \text{and using} \quad X(t)=A_0\cos(\omega_0 t+\varphi_0) & X(0)=2a=A_0\cos\varphi_0 \\ \dot{X}(0)=0 & \text{and using} \quad \dot{X}(t)=-\omega_0 A_0\sin(\omega_0 t+\varphi_0) & \dot{X}(0)=0=-\,\omega A_0\sin\varphi_0 \end{array}\right\} \qquad (12)$$

The second equation demands, for $A_0\neq 0$, that we choose $\varphi_0=0$, giving, from the first equation, $A_0=2a$. The result is that we have excited only the lowest normal mode,

$$\begin{array}{lcl} X(t)=2a\cos\omega_0 t & & x(t)=\tfrac{1}{2}(X+Y)=a\cos\omega_0 t \\ & \text{giving particle displacements} & \\ Y(t)=0 & & y(t)=\tfrac{1}{2}(X-Y)=a\cos\omega_0 t \end{array} \qquad (13)$$

It's now apparent why we chose 2*a* rather than *a* for the initial value of *X*!

Notice that the two masses oscillate in synchrony (ie. exactly in phase) with *x*(*t*) = *y*(*t*). To achieve this we initially displace them together to the right by the same distance *a*, hold them there and then release them together from rest at *t* = 0.

Clearly at *t* = 0 the coupling spring *s is not stretched at all*, and the equations tell us that it remains unstretched throughout the motion. We therefore understand why each mass will oscillate with frequency ω_0 independent of *s*: it's because *s* is never stretched or compressed and therefore plays no part in the motion; the two masses oscillate in unison as if there were no coupling at all.

Since these same equations describe the vertical oscillations of our slinky demonstration or the swings of two identical coupled pendulums, we don't have to repeat the analysis to see that this is also the way to excite the lowest modes of these systems: we move the two slinkys up by the same amount *a* and then let go, or we displace our pendulums by the same angle *a* in the same direction and let go.

(2) Exciting the highest mode, ω_1 with the normal coordinate Y: Since the normal coordinate for the other mode is now *X*, we have to choose initial conditions to ensure that $X(t) = 0$ at all times. This is accomplished by choosing

$$\begin{aligned} &X(0) = 0 \quad \text{and using} \quad X(t) = A_0 \cos(\omega_0 t + \varphi_0) \qquad X(0) = 0 = A_0 \cos\varphi_0 \\ &\dot{X}(0) = 0 \quad \text{and using} \quad \dot{X}(t) = -\omega_0 A_1 \sin(\omega_0 t + \varphi_0) \qquad \dot{X}(0) = 0 = -\omega_0 A_0 \sin\varphi_0 \end{aligned} \tag{14}$$

The first equation is solved by either $A_0 = 0$ or $\varphi_0 = \dfrac{\pi}{2}$; the second by $A_0 = 0$ or $\varphi_0 = 0$. Clearly the only possibility is therefore $A_0 = 0$ which ensures $X(t) = 0$ for all *t*.

Another way to think of this is to notice that, since $X = (x + y)/2$, the way to ensure that $X(t) = 0$ for all times is to ensure that $x(t) = -y(t)$ for all times, including *t* = 0 of course. The way to accomplish this is to set up initial conditions $x(0) = -y(0)$ and $\dot{x}(0) = -\dot{y}(0)$; this then ensures *X*(0) = 0 and at all subsequent times, ie. *X*(*t*) = 0. Thus, to excite the ω_1 mode all we need do is excite vibrations in *Y* alone by choosing an initial displacement $Y(0) = 2a$ and, for simplicity, an initial velocity $\dot{Y}(0) = 0$:

$$\begin{aligned} &Y(0) = 2a \quad \text{and using} \quad Y(t) = A_1 \cos(\omega_1 t + \varphi_1) \qquad Y(0) = 2a = A_1 \cos\varphi_1 \\ &\dot{Y}(0) = 0 \quad \text{and using} \quad \dot{\mathrm{Y}}(t) = -\omega_1 A_1 \sin(\omega_1 t + \varphi_1) \qquad \dot{Y}(0) = 0 = -\omega_1 A_1 \sin\varphi_1 \end{aligned} \tag{15}$$

The second equation demands, for $A_1 \neq 0$, that we choose $\varphi_1 = 0$, giving, from the first equation, $A_1 = 2a$. The result is that we have excited only the highest normal mode,

$$\begin{aligned} &Y(t) = 2a \cos\omega_1 t \qquad\qquad\qquad\qquad \mathrm{x(t)} = \tfrac{1}{2}(X + Y) = a\cos\omega_1 t \\ &\qquad\qquad\qquad \text{giving particle displacements} \\ &X(t) = 0 \qquad\qquad\qquad\qquad\qquad \mathrm{y(t)} = \tfrac{1}{2}(X - Y) = -a\cos\omega_1 t \end{aligned} \tag{16}$$

It's now apparent why we chose 2*a* rather than *a* for the initial value of *Y*!

Notice that the two masses oscillate in opposition (ie. exactly out of phase) with $x(t) = -y(t)$. To achieve this we initially displace them by equal distances, *a*, but in opposite directions, hold them there and then release them together from rest at *t* = 0. Clearly at *t* = 0 the coupling spring *s* is stretched as far as it ever will be, while one half-period later it is maximally compressed. We therefore understand qualitatively why the frequency ω_1 depends on *s* with a factor of 2: it's because *s* is being stretched and compressed by the two masses in co-operation to maximise its effect, ie. *s* is always doubly compressed or stretched.

Since these same equations describe the vertical oscillations of our slinky demonstration or the swings of two identical coupled pendulums, we don't have to repeat the analysis to see that this is also the way to excite the highest modes of these systems: we move one slinky up and the other down by the same amount *a* and then let go, or we displace our pendulums by the same angle *a* in opposite directions and let go.

(3) Exciting a linear combination of both normal modes, ω_0 and ω_1:

Having described how to excite pure normal mode vibrations, let's now explain the back-and-forth exchange of energy when we start the system off (at $t = 0$) *with only one mass displaced*. This is then not a normal mode. We first translate the initial conditions on x and y into initial conditions on the simpler normal co-ordinates X and Y:

$$x(0) = a \qquad y(0) = 0 \qquad\qquad X(0) = a \qquad Y(0) = a$$

or, for the normal co-ordinates, (17)

$$\dot{x}(0) = 0 \qquad \dot{y}(0) = 0 \qquad\qquad \dot{X}(0) = 0 \qquad \dot{Y}(0) = 0$$

With these initial conditions on the normal co-ordinates X and Y we can easily obtain the constants:

$X(0) = a$ and using $X(t) = A_0 \cos(\omega_0 t + \varphi_0)$ $\quad X(0) = a = A_0 \cos\varphi_0$

$\dot{X}(0) = 0$ and using $\dot{X}(t) = -\omega_0 A_0 \sin(\omega_0 t + \varphi_0)$ $\quad \dot{X}(0) = 0 = -\omega_0 A_0 \sin\varphi_0$ (18)

$Y(0) = a$ and using $Y(t) = A_1 \cos(\omega_1 t + \varphi_1)$ $\quad Y(0) = a = A_1 \cos\varphi_1$

$\dot{Y}(0) = 0$ and using $\dot{Y}(t) = -\omega_1 A_1 \sin(\omega_1 t + \varphi_1)$ $\quad \dot{Y}(0) = 0 = -\omega_1 A_1 \sin\varphi_1$ (19)

Again, as for single-particle SHM, these equations imply $\varphi_0 = \varphi_1 = 0$ and $A_0 = A_1 = a$.

The result is that we have excited a linear combination of normal modes:

$$X(t) = a\cos\omega_0 t \qquad x(t) = \tfrac{1}{2}(X + Y) = \frac{a}{2}(\cos\omega_0 t + \cos\omega_1 t)$$

and hence (20)

$$Y(t) = a\cos\omega_1 t \qquad y(t) = \tfrac{1}{2}(X - Y) = \frac{a}{2}(\cos\omega_0 t - \cos\omega_1 t)$$

[Note that we can recover our initial conditions by letting $t = 0$; $x(0) = a$ and $y(0) = 0$]

Thus we see that both normal modes of vibration have been excited and both masses oscillate in a superposition of these two modes. Notice that here neither mass executes SHM, whereas in the previous two cases both particles do indeed execute SHM: in case (1) with frequency ω_0 in case (2) with frequency ω_1. We can see what's involved and demonstrate explicitly the observed back-and-forth motion by recalling our previous treatment of BEATS: applying to the combinations in x and y the trig. addition formulae,

$$\cos A + \cos B = 2\cos\left(\frac{A - B}{2}\right)\cos\left(\frac{A + B}{2}\right)$$
$$\cos A - \cos B = 2\sin\left(\frac{A - B}{2}\right)\sin\left(\frac{A + B}{2}\right)$$

we discover that our solution reads,

$$x(t) = a\cos\left(\frac{\omega_1 - \omega_0}{2}t\right)\cos\left(\frac{\omega_1 + \omega_0}{2}t\right)$$
$$y(t) = a\sin\left(\frac{\omega_1 - \omega_0}{2}t\right)\sin\left(\frac{\omega_1 + \omega_0}{2}t\right)$$

Each displacement looks just like beats with our previously defined frequencies.
Note also that x and y are exactly in anti-phase, because one involves cosines, the other sines.
We illustrate with plots of $x(t)$ and $y(t)$ for weak coupling s.

.

Figure 7.3 Behaviour with time of individual masses, showing complete energy exchange in a time equal to half the beat period

Figure 7.4 Exciting pure and mixed modes.

General Method for Finding Normal Mode Frequencies.

In our simple 2 coupled oscillators example it was almost obvious how to find the NORMAL COORDINATES X, Y, with $X = x + y$, $Y = x - y$.

In more complicated cases lacking the symmetry of this problem, or with more than two coupled oscillators, it's not obvious how to find either the normal co-ordinates nor the corresponding frequencies. However there is a rather simple general method which is systematic and doesn't involve any guessing. It relies on the use of complex numbers. We will not use it, but basically the analysis indicates that N coupled oscillators have N coupled equations of motion, so the above technique for finding the normal mode frequencies requires eliminating N amplitudes, giving rise to an N^{th} order equation in ω^2 with N solutions. Thus, N coupled oscillators have N normal mode frequencies. For a continuum of oscillators, $N \to \infty$, there are therefore infinitely many normal mode frequencies!