Wrg-06 Addition of Vibrations

You need to know the following trigonometric identity.

$$\cos A + \cos B = 2\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

COMBINING OSCILLATIONS: Superposition

Sometimes two effects may compete to induce an oscillation at the same physical place. For example, as sound waves impinge on the ear, they make the eardrum oscillate at the frequency of the sound waves - i.e height of eardrum will go up and down $h(t) \propto \cos\omega t$. Suppose two different sound waves are hitting the ear drum at the same time - then the basic assumption is that the two vibrations simply *add* at each moment in time with resultant $h(t) = h_1(t) + h_2(t)$. What happens then depends on whether the vibrations are the same frequency, and whether they are in or out of phase.

In realistic physical situations a particle will usually be subjected to quite complicated disturbances. By Fourier's theorem these can be decomposed into a sum of harmonic ones, each of the form $A\cos(\omega t + \varphi)$. From the form of the SHM equation [Linearity: If x_1 and x_2 are solutions, so is $x = A_1 x_1 + A_2 x_2$. The resultant of 2 SHMs is just the sum of the two SHMs.

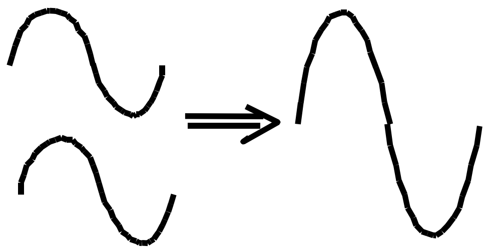
Examples to think about:

1) The vibrations of your ear drum subject to two harmonic waves of amplitude $A_1\cos(\omega_1 t + \varphi_1)$ and $A_2\cos(\omega_2 t + \varphi_2)$;

2) The resultant amplitude at a screen illuminated by monochromatic light from two slits;

3) Two simultaneous signals fed to the Y-plates of an oscilloscope.

We now study a few cases to illustrate the ideas.



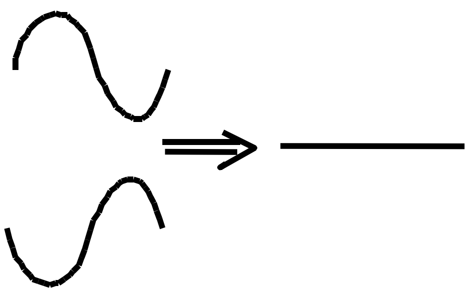
Ex(1)
Sum of two oscillations at same frequency ω, with equal amplitude A and unequal constant phases, φ_1 φ_2.

So $x_1 = A\cos(\omega t + \varphi_1)$ and $x_2 = A\cos(\omega t + \varphi_2)$

Therefore

$$x = x_1 + x_2 = A[\cos\omega t + \varphi_1) + (\cos\omega t + \varphi_2)]$$

$$x = 2A\cos\left(\frac{\varphi_2 - \varphi_1}{2}\right)\cos\left(\omega t + \frac{\varphi_1 + \varphi_2}{2}\right)$$



This is the same as SHM at frequency ω, with a constant amplitude of

$$A_{total} = 2A\cos\left(\frac{\varphi_2 - \varphi_1}{2}\right) \le 2A$$

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fig(a) For $\varphi_1 = \varphi_2$ we obtain $A_{total} = 2A$:
The two motions are exactly in phase and the amplitude is doubled. This is called **constructive interference.**

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fig(b) For $\varphi_1 - \varphi_2 = \pi$, $x = 0$, since the two are exactly out of phase. $[A_{total} = 2A\cos\frac{\pi}{2} = 0]$.
The oscillations are 180° out of phase, i.e. in exact **antiphase**.

This is called **destructive interference**. If the two amplitudes are not equal, then the cancellation is not complete, and the net amplitude is $A = A_1 - A_2$ (When in phase we get $A = A_1 + A_2$ of course.)

Ex(2) Sum of two oscillations at different frequencies

The important feature here is that the phase relationship is not fixed - the two oscillations keep drifting in and out of phase with each other, so that for a while the amplitudes may be adding up, then after a while the opposite happens and they cancel out, and so on. This leads to the phenomenon of **beats**.

Let's keep life simple by assuming both oscillations have the same amplitude, but different frequencies, ω_1 and ω_2. $x_1 = A\cos(\omega_1 t + \varphi_1)$ and $x_2 = A\cos(\omega_2 t + \varphi_2)$
Using the same trig identity again, we see that

$$x = x_1 + x_2 = A\cos(\omega_1 t + \varphi_1) + A\cos(\omega_2 t + \varphi_2)$$

$$x = 2A\cos\left(\frac{\Delta\omega}{2}t + \frac{\Delta\varphi}{2}\right)\cos\left(\overline{\omega}t + \overline{\varphi}\right)$$

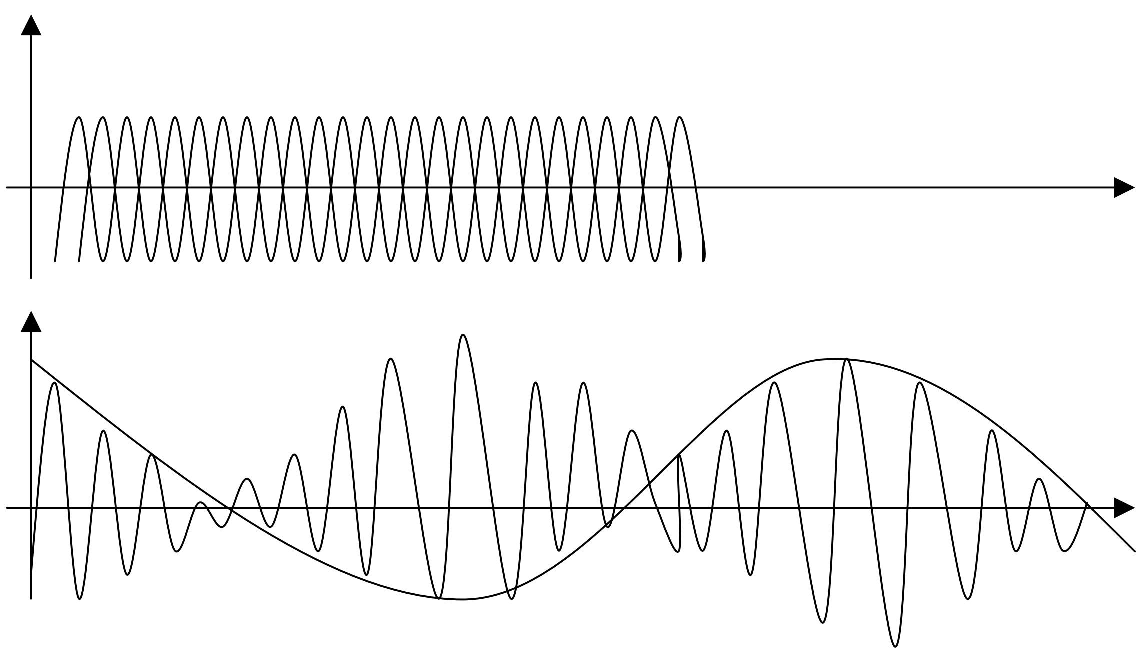
where

$$\Delta\omega = \omega_2 - \omega_1, \quad \overline{\omega} = \frac{\omega_1 + \omega_2}{2}$$

$$\Delta\varphi = \varphi_2 - \varphi_1, \quad \overline{\varphi} = \frac{\varphi_1 + \varphi_2}{2}$$

The $\overline{\omega}$ term by itself would be an oscillation at the average of the original two frequencies. The $\Delta\omega$ term by itself would be a slower oscillation (the frequency is smaller), caused by the drifting in and out of phase. As it stands the whole thing isn't strictly SHM at all. But if $\omega_1 \approx \omega_2$ the slow variation is so much slower that each cycle contains many cycles of the faster variation. Then it makes sense to think of the whole thing being like a simple SHM, $\cos\omega t$, whose amplitude varies slowly in time (amplitude $= 2A\cos\left[\frac{\Delta\omega}{2}t + \frac{\Delta\varphi}{2}\right]$). We say that the slower term **modulates** the SHM. We also sometimes call the slower variation an **envelope**. The whole phenomenon is called **beating**. With sound waves, the *note* we hear would correspond to the simple SHM at $\omega \approx \omega_1 \approx \omega_2$ whereas the slow term is heard as a periodic change in loudness.

Superposition of two oscillations



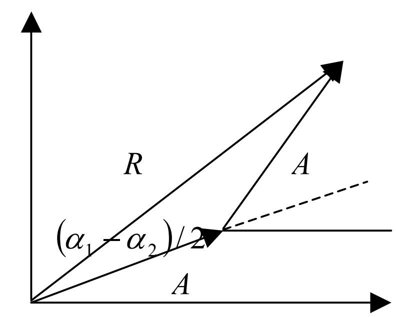
Note that successive maxima of beats come every half-period of $\cos\left(\frac{\omega_1 - \omega_2}{2}\right)t$.

So, the **beat period** is $$\boxed{T_{beat} = \frac{2\pi}{|\omega_1 - \omega_2|}}$$

Addition of vibrations using complex numbers

We have seen how to add vibrations using cos and sin, but it can be done easier with complex numbers. We will illustrate this with the addition of only two vibrations, but it is easily extended to the addition of many waves, such as in scattering from many slits.

Consider again the addition $x = x_1 + x_2 = A\cos(\omega_1 t + \varphi_1) + A\cos(\omega_2 t + \varphi_2)$



Let $z_1 = Ae^{i(w_1 t + \varphi_1)} = Ae^{i\alpha_1}$ and $z_2 = Ae^{i(w_2 t + \varphi_2)} = Ae^{i\alpha_2}$

Resultant is $z = z_1 + z_2 = Ae^{i\alpha_1} + Ae^{i\alpha_2}$

$$\begin{aligned} R^2 = ZZ^* &= A^2(e^{i\alpha_1} + e^{i\alpha_2})(e^{-i\alpha_1} + e^{-i\alpha_2}) \\ &= A^2(1 + e^{i(\alpha_1 - \alpha_2)} + 1 + e^{-i(\alpha_1 - \alpha_2)}) \\ &= 2A^2(1 + \cos(\alpha_2 - \alpha_1)) = 2A^2(1 + 2\cos^2\left(\frac{\alpha_2 - \alpha_1}{2}\right) - 1) \\ &= 4A^2\cos^2\left(\frac{\alpha_2 - \alpha_1}{2}\right) \end{aligned}$$

Hence

$$x = \operatorname{Re} z = 2A\cos\frac{(\alpha_2 - \alpha_1)}{2}\cos\left(\frac{\alpha_1 + \alpha_2}{2}\right)$$

$$x = 2A\cos\left(\frac{\Delta\omega}{2}t + \frac{\Delta\varphi}{2}\right)\cos(\overline{\omega}t + \overline{\varphi})$$ as before.

This is no easier than using direct addition of cosines, but this new method gains as the number of additions increases.

Sum of two oscillations in different directions

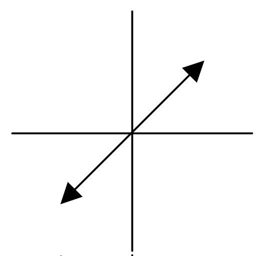
Instead of just adding the two vibrations in line, we look at the vector sum. The net result is a two-dimensional vibration. This kind of thing often occurs in all sorts of Engineering problems of course, but it's also important for basic physics, such as helping us understand polarisation effects in light. Simple light waves oscillate in a single direction, perpendicular to the direction of propagation of the light. However real beams of light are often made up of lots of waves vibrating in different directions, mingled together. The physics of light waves will become clearer later. For the moment let's get the basic idea of 2D oscillations -

You can do this as follows. Hang a weight on a string. Think of the floor as the xy plane. Pull the weight out in one direction - call this direction x. But instead of just letting it go as for usual pendulum, push off in the perpendicular direction (y). The pendulum traces out a repeating curved path, i.e. a 2D oscillation. What determines the curve? If one oscillation is $x(t)$ say, and the other is $y(t)$, the basic trick is to eliminate t and end up with an (x,y) equation, i.e. the equation of the curve in the (x,y) plane.

We'll get the basic results if we keep life simple and consider two oscillations with the same frequency, same amplitude, and oscillating at right angles. So we shall just concentrate on the phase difference, δ .

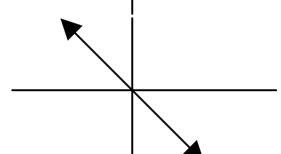
$\delta = 0 \qquad x = A\cos\omega t \qquad y = A\cos\omega t$

$\Rightarrow \qquad y = x$ i.e diagonal straight line



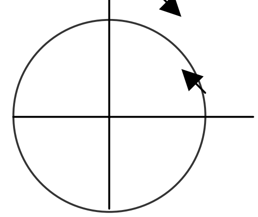
$\delta = \pi \qquad x = A\cos\omega t \qquad y = A\cos(\omega t + \pi) = -A\cos\omega t$

$\Rightarrow \qquad y = -x$ opposite diagonal



$\delta = \pi/2 \qquad x = A\cos\omega t \qquad y = A\cos(\omega t + \pi/2) = -A\sin\omega t$

$\Rightarrow \qquad x^2 + y^2 = A^2$ circle of radius A



Arbitrary δ

A bit messier, but not too tricky -

Use the following trig identity: $\boxed{\cos(\theta+\varphi)=\cos\theta\cos\varphi-\sin\theta\sin\varphi}$

$x = A\cos\omega t \qquad y = A\cos(\omega t+\delta) = A\cos\omega t\cos\delta - A\sin\omega t\sin\delta$

so $\quad y - x\cos\delta = A\cos\omega t\cos\delta - A\sin\omega t\sin\delta - A\cos\omega t\cos\delta = -A\sin\omega t\sin\delta$

$$y^2 + x^2\cos^2\delta - 2xy\cos\delta = A^2\sin^2\omega t\sin^2\delta$$

but $\quad \sin^2\omega t = 1-\cos^2\omega t = 1-\dfrac{x^2}{A^2}$, so RHS of above gives $A^2\sin^2\delta - x^2\sin^2\delta$.

Hence $\quad y^2 + x^2(\cos^2\delta+\sin^2\delta) - 2xy\cos\delta = A^2\sin^2\delta$

and finally the equation of the curve is

$x^2 + y^2 - 2xy\cos\delta = A^2\sin^2\delta$

This is the equation of an ellipse, tilted with respect to the xy axes

In the study of light, one would find that:

$\delta = 0$ or π cases correspond to **plane polarisation**
$\delta = \pi/2$ or $3\pi/2$ cases correspond to **circular polarisation.**