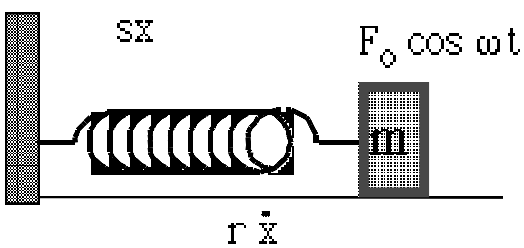
Wrg-05 Damped, Driven SHM & Resonance

To maintain the oscillations of a damped oscillator, we need to supply energy to replace that lost by the $e^{-\gamma t}$ decay.

Suppose, instead of just letting something vibrate at its natural frequency, we give it a nudge and try to drive it at our chosen frequency by applying this periodic force. This driving force is either pure sinusoidal, $F_0 \cos \omega t$, or a linear combination of different frequencies (Fourier). $F = \sum_i F_i \cos \omega_i t$

For example: an electrical oscillation drives a loudspeaker cone, the vibrations of the cone force vibrations in the air, sound waves hitting the walls of the room force vibrations in the walls, and the walls drive the air in your neighbours' room! We notice that the bass notes get through the wall rather more efficiently than the treble notes. These are the notes that are closer to the natural frequency of the wall.

Consider a mass attached to a spring, but now subject to a driving force.



equilibrium position

The equation of motion now has the additional driving force term.

So if we try to drive the mass-spring system at frequency ω, we have

$$m\ddot{x} = -r\dot{x} - sx + F_0 \cos \omega t$$

The general equation for forced oscillations is then

$$\boxed{\ddot{x} + \gamma \dot{x} + \omega_0^2 x = f_0 \cos \omega t}$$

where for the mass-spring system, $\omega_0^2 = s/m$, $\gamma = r/m$ and $f_o = F_0/m$.

Observation of forced vibrations indicates that

1) Things do vibrate at the chosen frequency, although the amplitude obtained depends on the driving force frequency, and is largest near the natural frequency.
2) Less obvious, but true, the resulting vibrations lag behind the driving force.

We now wish to solve this differential equation.
Begin by considering the complex variable z that obeys the equation

$$\boxed{\ddot{z} + \gamma \dot{z} + \omega_0^2 z = f_0 \cos \omega t}$$

The solution we seek is the real part of the solution of the above equation. $x = \mathrm{Re}(z)$.

As before consider a solution of the form $z = Ae^{i(pt+\varphi)}$ where A and φ have real values, but may depend on ω.
This will allow us to calculate the amplitude and phase variation with time.

For this to be a solution we require

$$\left(-p^2 + ip\gamma + \omega_0^2\right)Ae^{i\varphi}e^{ipt} = f_0 e^{i\omega t}$$

For the time dependence on both sides to be the same, we see immediately that $\boxed{p = \omega}$.

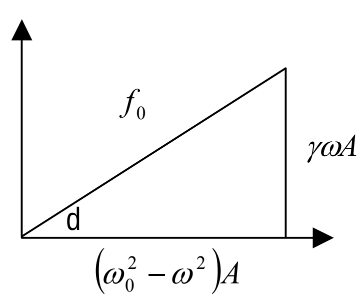
Now cancel $e^{i\omega t}$ and multiply by $e^{-i\varphi}$. For convenience later make the name change $\boxed{\delta = -\varphi}$.

We obtain

$$\left(\omega_0^2 - \omega^2\right)A + i(\gamma\omega)A = f_0 e^{i\delta}.$$

We now solve this equation for the amplitude, A and the phase lag, δ.

The simplest way to solve is to represent the 3 terms as vectors on an Argand diagram.
The equation implies the real term $\left(\omega_0^2 - \omega^2\right)A = \quad \Rightarrow\Rightarrow\Rightarrow$ in the horizontal, with $i(\gamma\omega)A = \quad \Uparrow$ in the vertical.



To complete the construction of the triangle

$\left(\omega_0^2 - \omega^2\right)A + i(\gamma\omega)A = f_0 \cos\delta + if_0 \sin\delta$ so
$\left(\omega_0^2 - \omega^2\right)A = f_0 \cos\delta$ and $(\gamma\omega)A = f_0 \sin\delta$

This right-angled triangle representing the vector sum can now be solved for the amplitude, A and the phase lag, δ:

$$\left(\omega_0^2 - \omega^2\right)^2 A^2 + (\gamma\omega)^2 A^2 = f_0^2$$

$$\boxed{A = \frac{f_0}{D^{\frac{1}{2}}} \qquad \text{where} \quad D \equiv \left(\omega_0^2 - \omega^2\right)^2 + (\gamma\omega)^2}$$

and

$$\sin\delta = \frac{\gamma\omega A}{f_0} = \frac{\gamma\omega}{f_0}\left(\frac{f_0}{D^{\frac{1}{2}}}\right) = \frac{\gamma\omega}{D^{\frac{1}{2}}}$$

$$\cos\delta = \frac{\left(\omega_0^2 - \omega^2\right)A}{f_0} = \frac{\left(\omega_0^2 - \omega^2\right)}{D^{\frac{1}{2}}}$$

where we have used the value of A to simplify.

The final solution is $z = Ae^{i(\omega t - \delta)}$ and hence $x = \mathrm{Re}(z) = A\cos(\omega t - \delta)$.

So $$x(t) = \frac{f_0}{D^{\frac{1}{2}}}\cos(\omega t - \delta).$$

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Properties of the **amplitude** and the **phase**.

Amplitude: A study of the amplitude leads to a resonance phenomenon that is one of the most ubiquitous and important in all of science and engineering.

The output amplitude depends on both the driving frequency and the damping factor. Let us see where the maximum amplitude or resonance occurs -

$$\boxed{A = \frac{f_0}{\left(\left(\omega_0^2 - \omega^2\right)^2 + (\gamma\omega)^2\right)^{\frac{1}{2}}}}$$

Clearly this is max when bottom line is min.

That is when $D = \left(\omega_0^2 - \omega^2\right)^2 + (\gamma\omega)^2$ is a minimum.

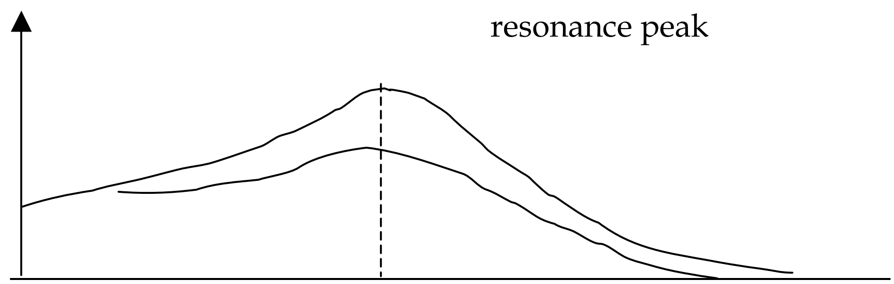
$\frac{dD}{d\omega} = 2\left(\omega_0^2 - \omega^2\right)(-2\omega) + 2\gamma^2\omega = 0$, for a maximum at $\omega = \omega_m = \sqrt{\left(\omega_0^2 - \frac{\gamma^2}{2}\right)}$.

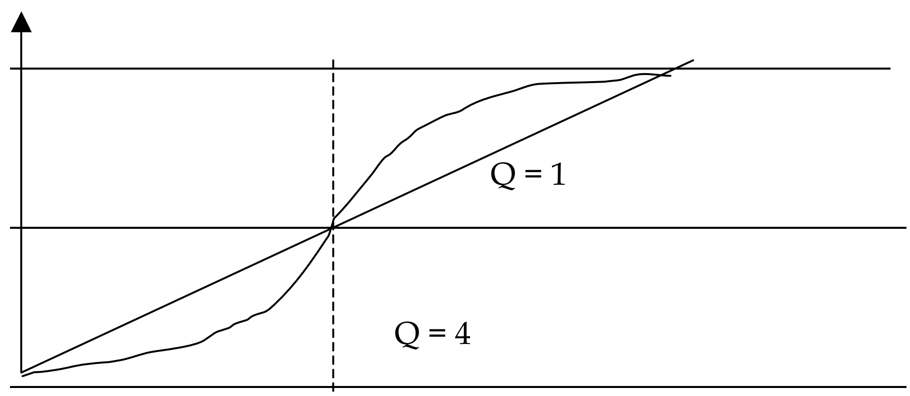
The maximum amplitude occurs at $A_{\max} \approx \frac{f_0}{\gamma\omega_0}$ for $\gamma << \omega_0$. The amplification factor is defined by comparison with the amplitude at low frequencies.

As $\omega \to 0, \quad A \to A_0 = \dfrac{f_0}{\omega_0^2}$ so the amplification factor is $\dfrac{A_{\max}}{A_0} \approx \dfrac{\omega_0}{\gamma}$. The lighter the damping, the greater the amplification.

Phase:

It is instructive to plot the resultant amplitude and phase lag as a function of frequency on the same diagram.





Comments

1) The motion $x(t)$ is the response of the system to a driving force $F_0 \cos \omega t$. This response has a resonance or enhanced amplitude, at the resonant frequency.
$\omega = \omega_m = \sqrt{\left(\omega_0^2 - \dfrac{\gamma^2}{2}\right)}$ where the resonance occurs at somewhat less than the natural frequency, but gets closer with smaller damping.

2) The sharpness of the resonance increases for smaller damping. There is a narrow width and rapid phase lag variation near the resonance.

3) The phase lag, δ, between the response and the driving force varies from $0 \to \dfrac{\pi}{2} \to \pi$ as ω varies from $0 \to \omega_m \to \infty$.

- At $\omega \approx 0$ the driving force varies slowly. Therefore the system has time to respond to the applied force, giving $\delta \sim 0$ in phase with the motion.
- As ω increases the system's response will lag further behind the increasingly rapidly varying driving force. (Due to $\gamma \neq 0$ $[\delta = 0 \text{ if } \gamma = 0]$ which makes the system sluggish).
  At very large ω they are exactly out of phase and $\delta = \pi = 180^0$.

- At resonance, $\delta = \pi/2 = 90^0$. This is a well-known criterion for resonance, particularly in nuclear and elementary particle physics. [Particle production varies with the energy available, peaking at the resonance energy for any particular particle.]

Power to drive the oscillator

* For SHM	Supply initial energy, which remains constant forever
* For damped SHM	Supply initial energy, but it dissipates Rate of energy loss = rate of doing work against friction
* Driven oscillator	To keep constant amplitude against friction losses, must **continuously supply** energy (i.e. continuous **power** input)

(This had been hidden so far by assuming we can supply $F = F_0 \cos \omega t$ as required)

How much energy is needed depends on the amplitude of vibration that is required and so depends on the frequency at which we are trying to drive the oscillator. The power required is the rate at which the driving force is doing work, i.e. F d*x*/d*t*.

Power P = F$\dot{x}$ $= F_0 \cos \omega t \; \dot{x}$

$x = A\cos(\omega t - \delta)$ so $\dot{x} = -\omega A \sin(\omega t - \delta)$

so $P = -\omega F_0 A \cos \omega t \sin(\omega t - \delta) = -\omega F_0 A[\cos \omega t \sin \omega t \cos \delta - \cos^2 \omega t \sin \delta]$.

What we want is $\overline{P}$, the average over a cycle, i.e. $\frac{1}{T}\int_0^T P(t)\,dt$ where $T = \frac{2\pi}{\omega}$.

$$\int_0^T P dt = -\omega F_0 A \int_0^T [\cos\omega t \sin\omega t \cos\delta - \cos^2\omega t \sin\delta] dt$$

$$= -\omega F_0 A \cos\delta \int_0^T [\cos\omega t \sin\omega t] dt + \omega F_0 A \sin\delta \int_0^T [\cos^2\omega t] dt$$

$$= F_0 A \sin\delta \int_0^T [\tfrac{1}{2}(1+\cos 2\omega t)]\omega dt = F_0 A \sin\delta \int_0^{\omega T = 2\pi} [\tfrac{1}{2}(1+\cos 2\theta)] d\theta \quad \text{where} \quad \theta = \omega t.$$

$$= F_0 A \sin\delta \left(\tfrac{1}{2}\left[\omega T - 0\right]\right).$$

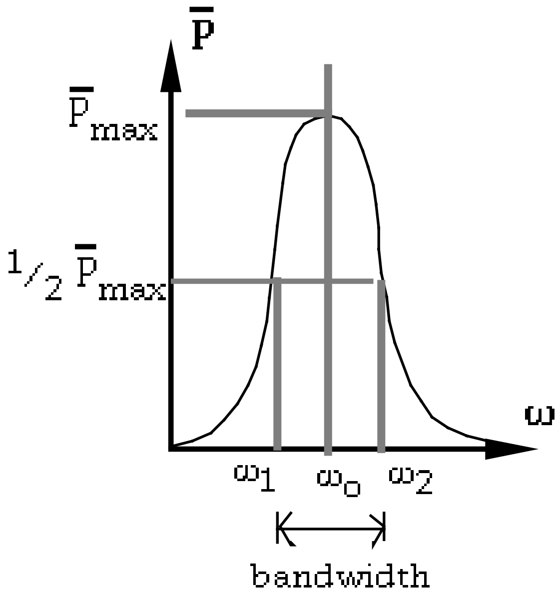
So $\overline{P} = \frac{1}{2}\omega F_0 A \sin\delta$ and $A = \frac{f_0}{\left(\left(\omega_0^2 - \omega^2\right)^2 + (\gamma\omega)^2\right)^{\frac{1}{2}}}$ and putting in for $\sin\delta = \frac{\gamma\omega}{D^{\frac{1}{2}}}$.

$$\overline{P} = \frac{1}{2}\,\frac{m f_0^2 \omega^2 \gamma}{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}$$

This is the time-averaged power required to keep constant amplitude, at a particular frequency. Or, assuming power can be supplied as needed, this is the power absorbed by the oscillator.

Power Resonance

Let's look at the shape of the **power curve**. Dividing top and bottom by ω^2.



$$\overline{P} = \frac{1}{2}\;\frac{mf_0^2\gamma}{\gamma^2 + \dfrac{(\omega_0^2-\omega^2)^2}{\omega^2}}$$

This is simpler than the amplitude curve. Except for large values of δ, the curve is fairly symmetrical. It tends to zero for both very small and very large ω. From inspection of the formula, you can see that the maximum will occur at $\omega = \omega_0$, the natural frequency of the oscillator, regardless of the strength of the damping.

On the other hand, the size of the power maximum does depend on damping.

Power resonance occurs at	$\omega = \omega_0$
Power at resonance maximum is	$\overline{P}(\max) = \dfrac{mf_0^2}{2\gamma}$

Sharpness of Resonance and Bandwidth

Above I drew only one power curve, but just as with the amplitude curve, the smaller the damping, the "sharper" the curve in some sense. We need a way to formalise and quantify the *width* of the power curve. The most common way to do this is to define a quantity called **bandwidth**, the distance between the **half-power points.** This has a simple formula as long as we stick to light damping.

$$\overline{P}_{\frac{1}{2}} = \frac{1}{2}\overline{P}(\max) \qquad \text{when} \qquad \frac{1}{2}\;\frac{mf_0^2\gamma\;\omega^2}{\gamma^2\omega^2 + (\omega_0^2-\omega^2)^2} = \frac{mf_0^2}{4\gamma}\,.$$

So $\quad 2\gamma^2\omega^2 = \gamma^2\omega^2 + \left(\omega_0^2 - \omega^2\right)^2$

$$\gamma^2\omega^2 = \left(\omega_0^2 - \omega^2\right)^2$$

$$\gamma\omega = \pm\left(\omega_0^2 - \omega^2\right)$$

The two values of ω corresponding to the + and - solutions are ω_1 and ω_2.

To get ω_1 we take the + version, and find

$\omega_1^2 + \gamma\omega_1 - \omega_0^2 = 0$ and so $\omega_1 = \dfrac{-\gamma \pm \sqrt{(\gamma^2 + 4\omega_0^2)}}{2}$.

Because we are assuming light damping, $\gamma << \omega_0$, so this reduces to $-\gamma/2 \pm \omega_0$.
We ignore the solution with $-\omega_0$ as unphysical, so finally we have

$$\omega_1 = \omega_0 - \gamma/2$$

Similarly for ω_2 we take the - version, and get

$\omega_2^2 - \gamma\omega_2 - \omega_0^2 = 0$ and then, with $\gamma \langle\langle \omega_0$, we find

$$\omega_2 = \omega_0 + \gamma/2$$

So the bandwidth is $\boxed{\omega_2 - \omega_1 = \gamma}$.

$$\boxed{\text{Bandwidth} = \gamma}$$

Therefore a good measure of *sharpness* is then the *inverse* of the bandwidth, but to put this on an absolute scale so that we can compare one oscillator to another, we should scale it to the resonance frequency:

$$\boxed{Sharpness = \frac{\omega_0}{bandwidth} = \frac{\omega_0}{\gamma}.}$$

The Quality of an oscillator

A lightly damped vibrating system can be thought of as *high quality* in that, when left to vibrate by itself, it will *ring* for a long time, and when forced it is *well tuned*, i.e. the resonance peak is sharp, and the amplification factor large. Some Physics and Engineering texts formally define a quantity called the **Quality Factor.** Unfortunately the precise definitions one can encounter here and there all differ slightly! However, for *light damping*, all the definitions are approximately equal to $\dfrac{\omega_0}{\gamma}$, so we will take that as our definition for this course.

So for a forced oscillator, Quality Q $= \dfrac{\omega_0}{bandwidth} = \dfrac{\omega_0}{\gamma}$ = sharpness of oscillator,

and $\dfrac{A_{max}}{A_o}$ = amplitude amplification factor as seen earlier.

Qualities of some real Oscillators in Nature

To get a feeling for Q, we can quote some numbers. Mechanical vibrations in engine parts and so on might have Q values of a few tens. High quality mechanical oscillators, like musical instruments vibrating in air, have Q values of thousands. Radio tuners, where an electrical circuit resonates when driven by an incoming electromagnetic wave, have Q ~ 100-1000. Atoms can also be seen as resonant oscillators, that can be driven by incoming light. They have a set of characteristic frequencies (which can be deduced from quantum theory), and will absorb power from light beams in very narrow ranges of frequency. This is the cause of the **dark Fraunhofer lines** in the Sun's spectrum. The lines are very narrow, corresponding to Q ~ 10^{24}.

Four important frequencies

It's important to keep in mind four different frequencies - I have tried to separate them by consistent symbols as we go along.

ω_0 is the natural frequency at which a system will vibrate if undamped and unforced.

ω_d is the damped frequency, at which a system will vibrate if damped but not forced.

ω_m is the frequency at which the amplitude maximum occurs in resonance.

ω is the forcing frequency, i.e. it is a frequency we choose to drive an oscillator at.