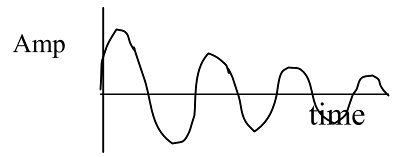
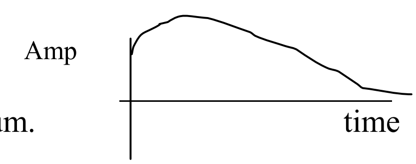
Wrg-04 Damped Vibrations

The ideal oscillator has constant energy, and goes on forever. In the real world, energy always gets dissipated, e.g. by friction or by electrical resistance, or by viscosity. Generally speaking the energy lost to such frictional forces ends up as heat somewhere. The loss of energy means that the amplitude of the oscillator dies away. If a pendulum, for example, starts to swing from an initial amplitude, or displacement, then the friction against the air (and also at the suspension point) will gradually brake, or slow down, the pendulum. This produces **damped harmonic motion.**



A plot of the decaying oscillations indicates an oscillatory term multiplied by an exponential decay term, as the mathematical solution.

As the strength of the friction increases, the oscillations die away more and more rapidly.

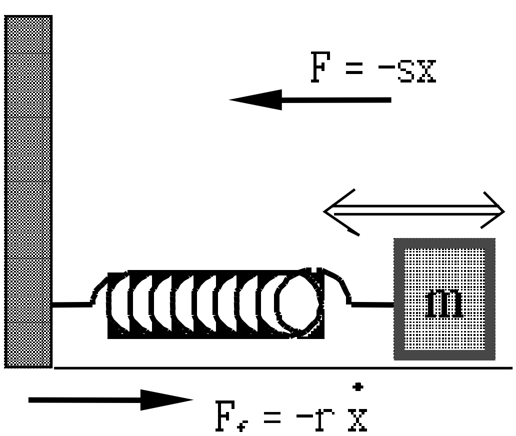


Past some **critical amount of damping**, there are no oscillations at all, but only a slow decay back to equilibrium.

A key point is that the frictional force is often proportional to the velocity, and acts in the opposite direction to the velocity, slowing down the motion.

$$F_{frict} \propto -\frac{dx}{dt}.$$

This applies to frictional effects experienced by a body moving through a gas or liquid with speeds small enough that turbulence or shocks caused by supersonic motion are not significant. This is the case for example for a pendulum swinging in some sort of fluid, or even in air, and for a mass-spring system on a rough table, when dynamical friction is proportional to velocity.



For a spring on a rough table, if the **coefficient of friction** is *r*, then the frictional force is

$$F_{frict} = -r\frac{dx}{dt}.$$

To get the equation of motion, we use $F = ma,$ but now we include the restoring force and the frictional force. So

$$F = ma = m\frac{d^2x}{dt^2} = -r\frac{dx}{dt} - sx$$

Many systems approximate well to this form of equation. By analogy to the general equation for SHM, the **general equation for damped SHM** is

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0}$$

where $$\boxed{\gamma = \frac{r}{m} \quad \text{and} \quad \omega_0^2 = \frac{s}{m}}$$

Note on viscosity η:
For an object of mass *m* and diameter *d* moving through a fluid of viscosity η at velocity, v, Stokes Law tells us that the viscous drag force is $F_{drag} = -3\pi d\eta v = -(3\pi d\eta)v$. So an oscillator in a fluid has a frictional constant $\gamma = {3\pi d\eta}/{m}$.

Note on ω_0:
Whenever the physics leads to equations of the above form, we get damped SHM. Remember that ω_0 is the natural angular frequency that the oscillator would have in the absence of damping, (i.e. when $\gamma = 0$). We write ω with the subscript "0" because it turns out that the oscillations we get in a damped situation are at a different frequency to that of the natural frequency.

Solution of the General Damped SHM Equation $\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0}$

There are several ways to solve this equation. Let us initially make a guess at the solution. From what we *observe* to happen experimentally:

(1) We see oscillations, but not necessarily at ω_0 - so guess a term with the form $\cos\omega t$
(2) We see a decay of the amplitude, and guess the decay is exponential - so use a term with the form

e^{-pt} where *p* is a constant.

So we try a solution of the form $x(t) = A_0 e^{-pt}\cos\omega t$.

We write "A_0" rather than just "A" because we can see that the amplitude varies with time and this is the value at time zero. We don't yet know the values of ω and *p*; we could find them by substituting our guessed solution into the differential equation and seeing what values they must take for the solution to work.

A simpler way is to solve the complex version of the equation: $\ddot{z} + \gamma\dot{z} + \omega_0^2 z = 0$.

Try a solution of the form $z = A_0 e^{i(pt+\varphi)}$ where A_0 and φ are real constants. Note that p can be complex. This form allows for an exponential amplitude decay multiplied by an oscillatory part. This is a similar form to that used to solve the SHM equation, and also agrees with the suggested solution above.

One can picture z as a vector in 2-dim, with x as its projection onto the x-axis.

Note that
$$z = A_0 e^{i(pt+\varphi)}$$
$$\dot{z} = ipA_0 e^{i(pt+\varphi)}$$
$$\ddot{z} = -p^2 A_0 e^{i(pt+\varphi)}$$

Substituting, and demanding that this be a solution for $A_0 \neq 0$:

$$\left(-p^2 + ip\gamma + \omega_0^2\right)A_0 e^{i(pt+\varphi)} = 0 .$$

As this equation is *true for all time*, one obtains a quadratic relationship for p,

$$p^2 - i\gamma p - \omega_0^2 = 0 \ .$$

Therefore $$p = \frac{+i\gamma \pm \sqrt{\left(-\gamma^2 + 4\omega_0^2\right)}}{2}$$ where $i^2 = -1$.

Hence
$$p = i\left(\frac{\gamma}{2}\right) \pm \sqrt{-\left(\frac{\gamma}{2}\right)^2 + \omega_0^2}$$
$$p = i\left(\frac{\gamma}{2}\right) \pm \omega_d$$
where $$\boxed{\omega_d^2 = \omega_0^2 - \frac{\gamma^2}{4}}$$

I use the term ω_d to show that this is the damped frequency, which is different to that of the natural frequency. ω_d is smaller than ω_0, and also depends on the magnitude of the damping.

So $$z = e^{i(pt+\varphi)} = A_0 e^{-\frac{\gamma}{2}t} e^{i(\pm\omega_d t+\varphi)}$$

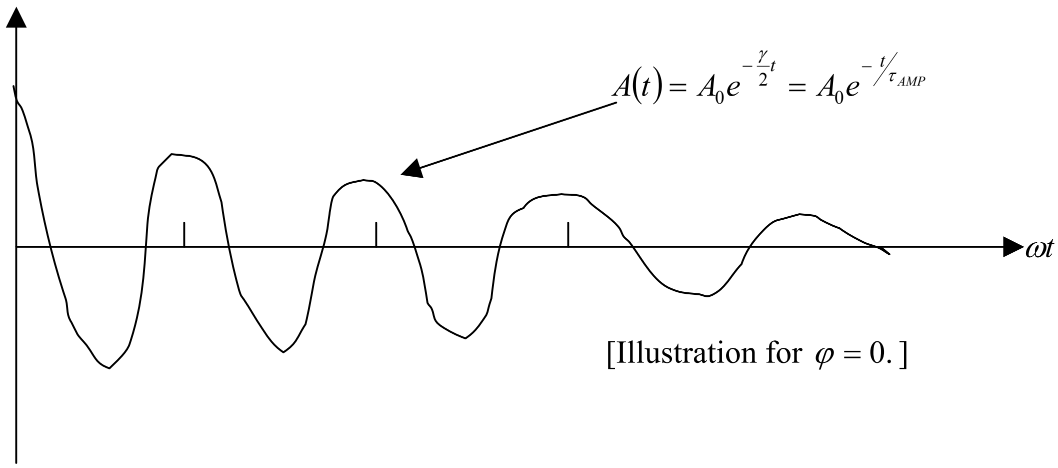
This gives, using $+\omega_d$ solution $$\boxed{x = \mathrm{Re}(z) = A_0 e^{-\frac{\gamma}{2}t} \cos(\omega_d t + \varphi) = A(t)\cos(\omega_d t + \varphi)}$$

Note: Why did I ignore the $-\omega_d$ possibility?

The reason is that it would give the term $e^{-i(\omega_d t - \varphi)}$. Check this.

And hence the solution $x = A_0 e^{-\frac{\gamma}{2}t} \cos(\omega_d t - \varphi)$

Since φ is arbitrary at this stage, its sign is not important, and this form is the same as our solution with $+\omega_d$.



[Illustration for $\varphi = 0$.]

The solution contains a term $A(t)$ which decays exponentially. It dies out to 1/e of its initial value (at t=o) in a time interval given by

$$\boxed{\Delta t = \tau_{AMP} = \frac{2}{\gamma}}$$

We can think of τ_{AMP} as the approximate lifetime of the motion, and write the amplitude as

$$A(t) = A_0 e^{-t/\tau_{AMP}}$$

This time interval is also called the **relaxation time**, or the **ringing time.** When damping is small the oscillations carry on for a long time, referred to as ringing.

Critical damping and overdamping

a) overdamping

We have so far assumed that the damping is small. Recall that $\omega_d^2 = \omega_0^2 - \frac{\gamma^2}{4}$, so we are assuming that $\omega_0^2 - \frac{\gamma^2}{4} > 0$. That is $\omega_0 > \frac{\gamma}{2}$.

But what if $\omega_0 < \frac{\gamma}{2}$? or $\gamma > 2\omega_0$

As damping increases the oscillations get slower, approaching zero frequency (infinite period) as γ approaches $2\omega_0$. If $\gamma > 2\omega_0$, the oscillations given by our formula are imaginary, so our guessed solution form does not work. What we observe in the world, as opposed to the maths, is that past a critical damping, there are no oscillations, only a decay.

So let us guess the following solution for the region beyond critical damping:

$$x = A_0 e^{-ct}$$

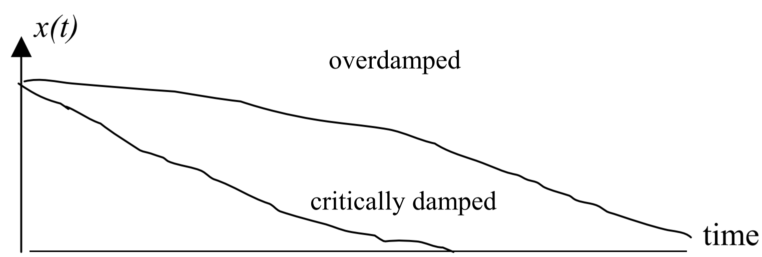
Now substitute into our original equation of motion.

$$A_0 e^{-ct}\left[c^2 - \gamma c + \omega_0^2\right] = 0$$
$$\text{So} \quad c^2 - \gamma c + \omega_0^2 = 0$$

The general solution for overdamped oscillator is

$$x = A_0 e^{-ct} \quad where \quad c = \frac{\gamma}{2} \pm \sqrt{\left[\gamma^2/4 - \omega_0^2\right]}$$

$$x(t) = \left(A_1 \mathrm{e}^{-\beta t} + A_2 \mathrm{e}^{+\beta t}\right)\mathrm{e}^{-\frac{\gamma}{2}t} \quad where \quad \beta = \sqrt{\left[\gamma^2/4 - \omega_0^2\right]}$$



So for an overdamped oscillator there are no oscillations, and the decay back to equilibrium is slower as damping increases.

b) critical damping

Consider the limiting case when $\boxed{\gamma = 2\omega_0}$

The return to equilibrium is quickest for this condition, called critical damping. In engineering situations, damping may be deliberately designed in, so that jolts to the system don't leave it vibrating, but on the other hand, don't take forever to sink back to equilibrium. What you want is the quickest return to normal - this is given by

$$\boxed{\text{Critical damping } \gamma = 2\omega_0}$$

The solution $x(t) = Ae^{-\frac{\gamma}{2}t}$ is not actually the most general solution, as it contains only one integration constant. It is in fact $x(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$. Try this for yourself.

Critical damping is designed into measuring equipment with moving parts (ammeters, voltmeters, galvanometers), vehicle suspension systems etc.

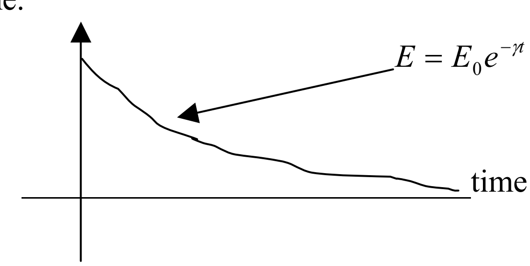
Energy of a damped oscillator

Recall that the energy of an ideal oscillator is $\frac{1}{2}m\omega_0^2 A^2$. This means that the energy of an oscillator is dissipated at a faster rate than the amplitude decreases.

Hence $$E = \tfrac{1}{2}m\omega_0^2 A^2 = \tfrac{1}{2}m\omega_0^2 A_0^2 e^{-\gamma t} = E_0 e^{-\gamma t}$$

Thus the energy decays away exponentally, (lost in friction), dying to 1/e of its initial value in the so-called relaxation time.

$$\boxed{\tau_{Energy} = 1/\gamma}$$



Q-values

A quantity used widely to parameterise real systems is the quality or Q-value.

$$Q = \frac{\text{Decay time scale for energy loss}}{\text{Free oscillation time scale}} = \frac{1/\gamma}{1/\omega_0}$$

So $Q = \dfrac{\omega_0}{\gamma}$ where Q measures the ability of the system to retain its initial energy.

There is another form for Q.
How many oscillations, n, occur in the time τ_{AMP} ?

$$n \approx \tau_{AMP} / T_0 = \frac{2/\gamma}{2\pi/\omega_0} = \frac{Q}{\pi}. \qquad \text{Therefore} \qquad Q \approx n\pi$$

That is Q measures approximate number of oscillations before amplitude decays to 1/e of its initial value.

Accounting for the energy loss.

Let us check, in the case of the mass-spring system, that the loss of energy makes physical sense.
For the mass-spring system $E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} sx^2$

The rate of energy decay is $\frac{dE}{dt} = m\dot{x}\ddot{x} + sx\dot{x} = m\dot{x}\left(\ddot{x} + \frac{s}{m}x\right)$

But $\ddot{x} + \frac{r}{m}\dot{x} + \frac{s}{m}x = 0$ from initial equation of motion.

Then $\frac{dE}{dt} = m\dot{x}\left(-\frac{r}{m}\dot{x}\right) = (-r\dot{x})\dot{x} = F_{friction}\dot{x}$

So Rate of energy decay of oscillator = Rate of doing work against frictional force.

{The work done ends up as heat.}