Wrg-03 Simple Harmonic Motion in Nature

(PHYSICAL OCCURRENCE OF SIMPLE HARMONIC MOTION)

Differential equation (4a), which leads to regular oscillations, tends to crop up when a system is disturbed from equilibrium, and there is a restoring force proportional to distance moved from equilibrium. The restoring force need not be directly proportional to this distance, but then the resulting vibrations are not so simple. Nonetheless many situations in nature are approximately **S**imple **H**armonic **M**otion with $F \propto -x$. An example of such harmonic motion is a simple mass-spring system. An outline of the physics and maths is given now, with more details given later. Physics tells us that when the **displacement** of the mass *m* from equilibrium is *x*, the force on the mass caused by the spring is *F(x) = -sx* where *s* is the "stiffness constant" of the spring. (The force is negative because it's directed back towards equilibrium). But we also know Newton's Second Law, *F = ma*, so

$$F(x) = -sx = m\frac{d^2x}{dt^2} \qquad \text{so} \qquad \frac{d^2x}{dt^2} = -\frac{s}{m}x.$$

Note that the acceleration associated with SHM is not constant (unlike the acceleration due to gravity).

The solution is $x = A\cos(\omega t + \varphi)$ where $\omega = \sqrt{\dfrac{s}{m}}$ and φ is the constant of integration.

SHM occurs whenever the physics leads us to an equation that looks like this mass-spring equation:

$$\frac{d^2x}{dt^2} = -\omega^2 x \quad \text{or more neatly} \quad \ddot{x} + \omega^2 x = 0$$

This then is a linear second order equation with constant coefficients.
It is the general equation for SHM, and has the general solution:

$$\boxed{x = A\cos(\omega t + \varphi)}$$

So displacement *x* changes with time periodically. The **period** of repetition is $T = \dfrac{2\pi}{\omega}$.

ω = **angular frequency** in radians/sec (physical property of the system)
$\theta = \omega t + \varphi$ = **phase angle** in radians (expresses position within a period)
A= maximum displacement or **Amplitude** (depends on initial conditions)
φ = initial phase or **phase constant** (depends on initial conditions)
ω is also called the **natural frequency** of vibration for the system in question.
We will often come across **linear frequency**, $f = \omega/2\pi$, which is in cycles/sec, or Hz.

The period of repetition then is $T = \frac{1}{f}$.

Example

An ultrasonic device used in medical diagnosis oscillates at a frequency of 7 MHz and has an amplitude of O.3 cm. Write down its displacement with time. What is the period of vibration?

$\omega = 2\pi f = 2\pi \times 7 \times 10^6 \quad \approx 4.4 \times 10^7$ rad/sec

So $\quad x = 0.3\cos(4.4 \times 10^7 t + \phi) \quad$ cm

$T = \frac{1}{f} = \frac{2\pi}{\omega} = \frac{2\pi}{4.4 \times 10^7} = 0.14\mu\text{s}$

Note that you cannot determine the phase angle ϕ from the given information.

[Of course $x = A\sin(\omega t + \varphi_2)$ would also be a solution to the SHM equation, but with the phase constant $\varphi_2 = \varphi + \pi/2$. (Check this.) So the two solutions are really the same thing seen at different times. We will normally stick to the *cos* version in these notes.

The **position** as a function of time is $\quad x = A\cos(\omega t + \varphi)$

The **velocity** as a function of time is $\quad \dot{x} = -A\omega\sin(\omega t + \varphi)$

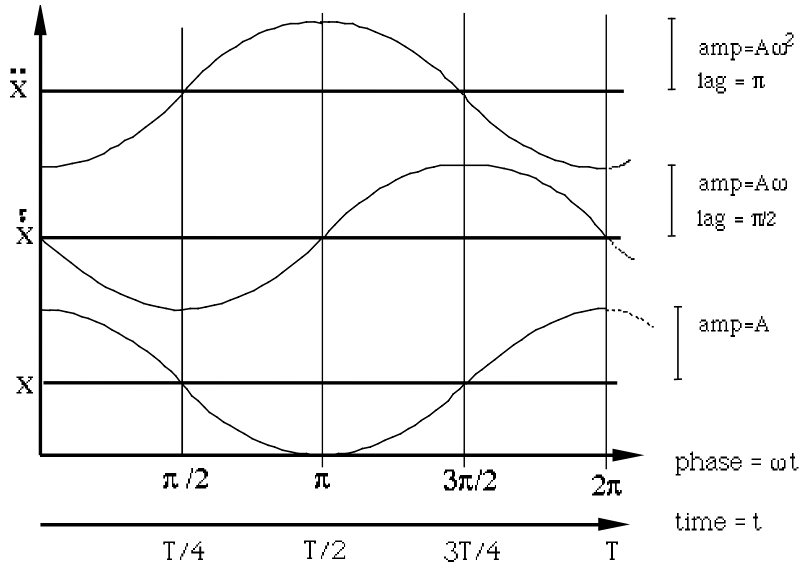
The **acceleration** as a function of time is $\quad \ddot{x} = -A\omega^2\cos(\omega t + \varphi)$

If we know what the *initial conditions* of position and velocity were, we can calculate what the resulting amplitude and phase constant will be:

at $t = 0$, $\quad x = x_0 \quad \dot{x} = v_0 \quad \Rightarrow\Rightarrow \quad x_0 = A\cos\varphi \quad v_0 = -A\omega\sin\varphi$

$$\Rightarrow\Rightarrow \varphi = \tan^{-1}\left(\frac{-v_0}{x_0\omega}\right) \quad \text{and} \quad A = \sqrt{\left(x_0^2 + \frac{v_0^2}{\omega^2}\right)}$$

Sketch of behaviour of *x(t)*, $\dot{x}$, *and* $\ddot{x}$ [Again for $t = 0$ $v_0 = 0, \quad \varphi = 0 \quad A = x_0$**]**



Example. A force of 6N, acting on a body B with mass 0.5kg, extends the spring by 0.03m.

Hence $s = -\frac{F}{x} = -\frac{6}{.03} = 200\text{N/m}$ ***and*** $\omega = \sqrt{\frac{s}{m}} = \sqrt{\frac{200}{0.5}} = 20 \text{ rads per sec}$.

The body is released from this position, with a velocity of 0.4m per sec towards the spring.

Find amplitude A and phase angle φ.

$$A = \sqrt{0.03^2 + \frac{0.40^2}{20^2}} = 0.036\text{m}$$

$$\varphi = \tan^{-1}\left(\frac{+v_0}{\omega\, x_0}\right) = +33.7^0 \quad \text{or} \quad +5.9 \text{ rad}$$

What is the maximum velocity and acceleration?

$$\dot{x}^2 = A^2\omega^2 \sin^2(\omega t + \varphi) = A^2\omega^2 - A^2\omega^2\cos^2(\omega t + \varphi) = \omega^2\left(A^2 - x^2\right)$$

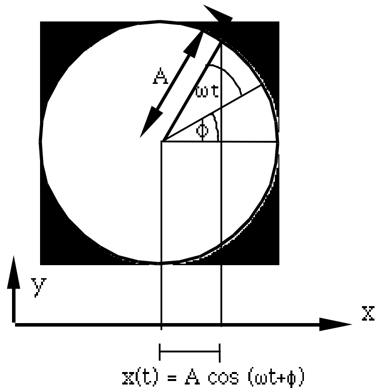
The velocity is max when x = 0, so

$$\dot{x}(\max) = 20 \times 0.036 = 0.72 \text{ ms}^{-1}$$

$$\ddot{x}(\max) = 20^2 \times 0.036 = 14.4\text{ms}^{-2}$$

Rotating vector picture (The vector is called a phasor).

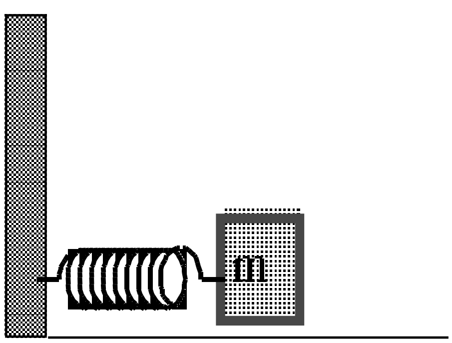
One can visualise oscillatory motion as the projection of uniform motion in an imaginary circle. The circular motion is in the (*x*,*y*) plane; the *x* co-ordinate is real, the *y* co-ordinate is purely imaginary for the purpose of playing this trick. Then the "phase angle" really is an angle, and it is easier to see the phase constant as a starting point. This way of visualising SHM helps when we want to understand the *difference* in phase between two oscillations.



For later, note that if z = x+iy is a point on this circle, then $x = |z|\cos\theta = \mathrm{Re}(z)$.

The general trick is to write down maths corresponding to the physics in question, then manipulate to make it look like the general SHM equation.

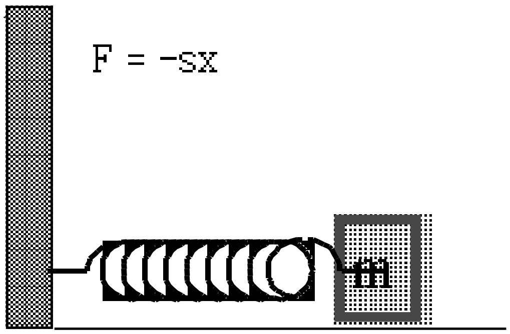
(1) A mass attached to a spring, supported on a table.



The simplest SHM is: a mass *m* oscillating from the end of a spring obeying Hooke's Law (stiffness s) and moving horizontally on a frictionless table. This diagram shows the spring neither extended nor stretched. The force on *m* is $F_x = 0$.

======> positive *x*-direction

$x = 0$ is the equilibrium position.



At time t the spring is extended by x. By Hooke's Law, tension in the spring is sx and acts against the extension in the negative x-direction.

Therefore the spring exerts a force on m, given by $F_x = -sx$.

By Newton's Second Law, the equation of motion for the mass is therefore

$$F_x = ma = m\frac{d^2x}{dt^2} = -sx$$

or $\frac{d^2x}{dt^2} = -\frac{s}{m}x$ Now define $\boxed{\omega^2 = \frac{s}{m}}$

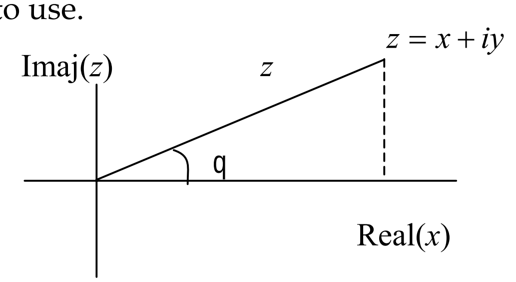
Hence the SHM equation is $\boxed{\frac{d^2x}{dt^2} = -\omega^2 x \quad \text{or} \quad \ddot{x} = -\omega^2 x}$

Solving the SHM equation:
I'll now solve this differential equation using complex numbers. In this case it looks like overkill, but later we will use the same method to solve more complicated equations with little or no increase in difficulty.

A quick revision of the complex algebra we need to use.

$$\begin{aligned} z &= x + iy \\ &= |z|\cos\theta + i|z|\sin\theta \\ &= |z|(\cos\theta + i\sin\theta) \\ &= |z|e^{i\theta} \end{aligned}$$

$i^2 = -1$



Firstly, write equation in complex form.

$$\boxed{\ddot{z} = -\omega^2 z}$$

and try a solution of the form $z = A_0 e^{i(pt+\varphi)}$ where A_0 and φ are real constants. Note that p can be complex. This form allows for an exponential amplitude decay multiplied by an oscillatory part.
Remember that $e^{i\theta} \equiv \cos\theta + i\sin\theta$

One can picture z as a vector in 2-dim, with x as its projection onto the x-axis.

Note that
$$z = A_0 e^{i(pt+\varphi)}$$
$$\dot{z} = ipA_0 e^{i(pt+\varphi)}$$
$$\ddot{z} = -p^2 A_0 e^{i(pt+\varphi)}$$

Substituting, and demanding that this be a solution for $A_0 \neq 0$:

$$\left(-p^2 + \omega^2\right) A_0 e^{i(pt+\varphi)} = 0,$$

As this equation is true for all time

$$p^2 = \omega^2$$

So $\quad z = A_0 e^{i(\varpi t+\varphi)} \quad$ and $\quad x = A_0\cos(\omega t + \varphi)$
We have solved a differential equation "without integrating" !

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We return now to the following solution which illustrates all the basic ideas most often encountered in Physics Differential equations in one variable. For the last time in this course, we will use the non complex method, with which you should be familiar. Solve $\ddot{x} = -\omega^2 x$.

(a) Since the equation is 2nd order, we need to integrate twice to find solution $x(t)$, and so we will have two constants of integration.

(b) We find that the solution can be written as a linear combination of two independent functions ($\sin\omega t, \cos\omega t$) each of which, on its own, is a solution.
$x = a\cos\omega t + b\sin\omega t \neq 0$ for all t, with a and b non zero.
$\equiv A\cos(\omega t + \varphi)$ incorporates the two functions in terms of one.

(c) The first integration gives *Conservation of Energy*. This is always the case when one integrates Newton's second law, provided no external forces are acting --- external forces add or subtract energy. See later.

In solving the SHM equation, $\ddot{x} = -\omega^2 x$, we use the common trick of separating the variables (x, and t). This cannot be done immediately, and requires a change of variable:

We replace t using $\dot{x} = \frac{dx}{dt}$ as follows

$\frac{d^2x}{dt^2} = \frac{d\dot{x}}{dt} = \frac{dx}{dt}\frac{d\dot{x}}{dx} = \dot{x}\frac{d\dot{x}}{dx}$ where we have used the chain rule of differentiation.

Hence the SHM equation becomes $\dot{x}\frac{d\dot{x}}{dx} = -\omega^2 x$ enabling us to separate $\dot{x}$ and x and to integrate, from t = 0 to t = t.

$$\int_{\dot{x}_0}^{\dot{x}(t)} \dot{x}\, d\dot{x} = -\omega^2 \int_{x_0}^{x(t)} x\, dx$$

$$\tfrac{1}{2}\dot{x}^2(t) - \tfrac{1}{2}\dot{x}_0^2 = -\tfrac{1}{2}\omega^2 x^2(t) + \tfrac{1}{2}\omega^2 x_0^2 \qquad \text{where } x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0$$

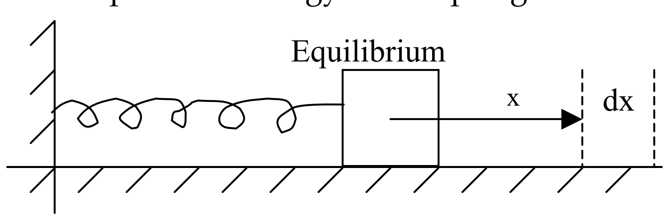
$$\tfrac{1}{2} m\dot{x}^2(t) + \tfrac{1}{2} m\omega^2 x^2(t) = \tfrac{1}{2} m\dot{x}_0 + \tfrac{1}{2} m\omega^2 x_0^2$$

This is a constant in time, equivalent to E. Hence $\boxed{\tfrac{1}{2} m\dot{x}^2(t) + \tfrac{1}{2} m\omega^2 x^2(t) = E}$.

We use the symbol E for the constant as the first term is the kinetic energy of the mass, and the second term is the potential energy of the spring.

Note that $\tfrac{1}{2} m\omega^2 x^2 = \tfrac{1}{2} m\frac{s}{m} x^2 = \tfrac{1}{2} s x^2$, the potential energy stored in the spring (say)

To show that $\tfrac{1}{2} s x^2$ is the potential energy in the spring:



Suppose the string is stretched by x from the equilibrium position, and then the spring is stretched a further infinitesimal amount dx.

The tension in the spring at x is $F_x = -sx$. So the work done in stretching the spring a distance dx is

$$dW = (sx)dx$$

Hence the total work done on the spring in stretching from 0 => x is the potential stored in the spring. $V(x) = \text{PE} = s\int_0^x x dx = \frac{1}{2}sx^2$.

We must integrate again and this time the equation is $\frac{1}{2}m\dot{x}^2(t) + \frac{1}{2}m\omega^2 x^2(t) = E$. Note we have introduced a constant of integration, E.

$$\text{Now } \frac{dx}{dt} = \dot{x} = \omega\sqrt{\left(\frac{2E}{m\omega^2} - x^2\right)}$$

Separating the variables and integrating we have

$\int \frac{dx}{\sqrt{A^2 - x^2}} = \omega\int dt = \omega t + \varphi$ where $A = \sqrt{\frac{2E}{m\omega^2}}$ is a known constant, and φ is the second constant of integration.

To integrate it is convenient to change variable. Let $x = A\cos\theta$.

Therefore $dx = -A\sin\theta \, d\theta = -A\sqrt{(1-\cos^2\theta)}\, d\theta = -\sqrt{\left(A^2 - x^2\right)}d\theta$

$$\int \frac{dx}{\sqrt{A^2 - x^2}} = -\int d\theta = -\theta = -\cos^{-1}\left(\frac{x}{A}\right)$$

$\cos^{-1}\left(\frac{x}{A}\right) = -(\omega t + \varphi)$ and so $x = A\cos(-[\omega t + \varphi]) = A\cos(\omega t + \varphi)$, giving us the general solution with two unknown constants.

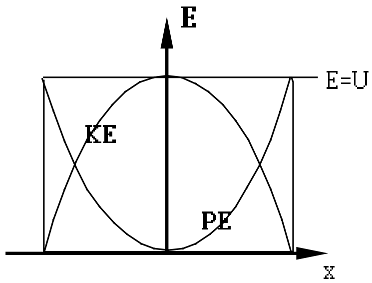
Energy changes in SHM. **Another view.**

Consider a mass m undergoing SHM with amplitude A and angular frequency ω.

Let's see how kinetic and potential energy vary round the cycle. At any given (x,t) :

$$\text{KE} = \tfrac{1}{2}m\dot{x}^2 = \tfrac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \varphi)$$

PE = work done against restoring force to reach x = $\frac{1}{2}sx^2 = \frac{1}{2}m\omega^2 A^2\cos^2(\omega t + \varphi)$



Total energy E(t) = KE(t) + PE(t)

$$= \tfrac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \varphi) + \tfrac{1}{2} m\omega^2 A^2 \cos^2(\omega t + \varphi)$$

$$= \tfrac{1}{2} m\omega^2 A^2$$

So energy swaps between KE and PE, but total energy is constant.

$$\boxed{E = \tfrac{1}{2} m\omega^2 A^2} \quad \text{or} \quad \boxed{E = \tfrac{1}{2} s A^2}$$

Notice that the potential energy is quadratic in x. Conversely, we can see that anything sitting in a potential well that goes as $V = sx^2$ will undergo SHM if disturbed. When displaced a small distance from the equilibrium, the force is given by the slope of the potential energy curve, $F = -dV/dx = -2sx$, which is of course just the force law required for SHM. Now any minimum looks roughly parabolic as long as you don't stray too far from the bottom ... (This can be shown mathematically with Taylor's theorem). This is why a lot of physical systems show approximate SHM as long as the amplitude of vibration is small. (e.g. the pendulum).

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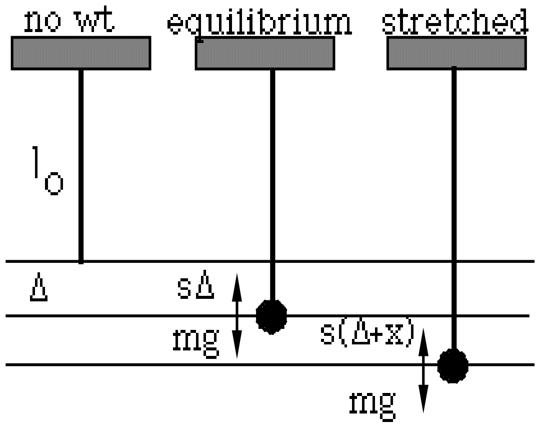
(2) Vertical oscillations of a mass suspended from an elastic string or a spring.

(a) **(b)** **(c)**



(a) Weight not connected. String not extended nor compressed.
(b) Equilibrium position. String tension balances gravitational force on m. Defines the $x = 0$ position.
(c) At time t, string tension acts upwards to restore mass to the equilibrium. Defines positive x value.

In equilibrium: String extended by Δ, producing a tension upwards of

$$F_x^{string} = -s\Delta\,.$$

This just balances the gravitational downward force on the mass, equal to

$$F_x^{gravity} = +mg.$$

Therefore the net downward force is $F_x = F_x^{string} + F_x^{gravity} = -s\Delta + mg = 0$ for equilibrium.

Hence $\boxed{s\Delta = mg}$.

Now the mass is set in motion and at time t has displacement x.
The total extension on the string is $(\Delta + x)$, and hence
$F_x^{string} = -s(\Delta + x)$ and $F_x^{gravity} = mg$.

The resultant downward force on the mass is
$F_x = F_x^{string} + F_x^{gravity} = -s(\Delta + x) + mg$

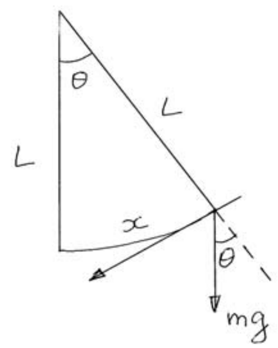
Hence $F_x = -sx = m\dfrac{d^2x}{dt^2}$ So $\dfrac{d^2x}{dt^2} = -\left(\dfrac{s}{m}x\right) = -\omega^2 x$ where $\omega = \sqrt{\dfrac{\mathrm{s}}{\mathrm{m}}}$.

This is SHM with $x(t) = A\cos(\omega t + \varphi)$.

Note: This is the same result as the horizontal oscillations of a mass attached to a spring ---- the effect of gravity has cancelled. The only difference between the horizontal and vertical spring cases is that in the vertical case the equilibrium position is at a slightly stretched position, due to gravity.

(3) Pendulum

Assume that the mass m is suspended by a massless, unstretchable string, of length L.



Approximately SHM - only accurate for small amplitude oscillations.

For small θ, restoring force is $mg\sin\theta$ and direction is approximately along x, so

$$F(x) = -mg\sin\theta, \quad \text{but} \quad \sin\theta \approx \frac{x}{L} \quad \text{so} \quad F(x) = -mgx/L$$

$$\ddot{x} + \left(\frac{g}{L}\right)x = 0$$

So, the natural frequency of a pendulum does not depend on the mass:

$$\boxed{\omega^2 = g/L}$$

Note that a pendulum can be used to measure g as T, L are well determined.
Local deposits of ore or oil may be indicated by local measurement variations in g.

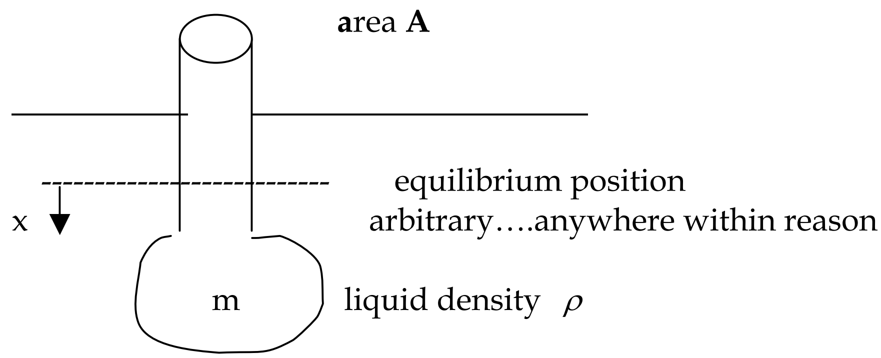
Example.

A simple pendulum of length 1.102m has a period of 2.21sec. What is the local value of g ?

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \quad \textbf{\textit{so}} \quad g = \frac{4\pi^2 L}{T^2} = \frac{4\pi^2 \times 1.102}{(2.21)^2} = 9.82 \text{ ms}^{-2}$$

(4) **Object bobbing in fluid**

For example a hydrometer or a ship afloat, bobbing up and down.
Assume a cylindrical object for simplicity
We need to use Archimedes' Principle, so look this up in your textbook.
It states that **The Buoyant Force on an immersed object equals the weight of the liquid displaced.**



When the hydrometer is pushed down by a distance x from the equilibrium position, it displaces more liquid than in the equilibrium position.

The extra mass of liquid displaced equals ρAx.
Therefore the extra force upwards on the hydrometer equals g times the mass of the extra liquid displaced $= g\rho Ax = -F_x$.

Applying Newton's second Law $\quad m\frac{d^2x}{dt^2} = F_x = -g\rho Ax \qquad$ m is mass of hydrometer

$$\Rightarrow \quad \ddot{x} = -\omega^2 x$$

So the natural frequency of a bouncing boat is given by

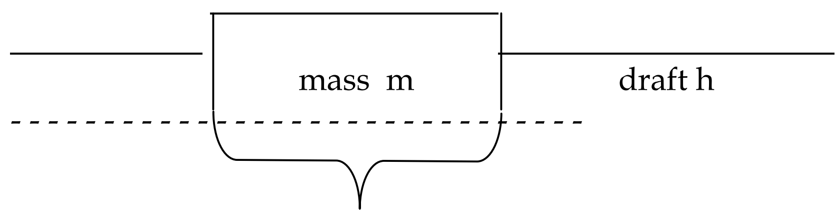
$$\boxed{\omega^2 = \frac{g\rho A}{m}}$$

Example 1) A battery hydrometer

Data
$m = 10g = 10^{-2}\,\text{kg}, \quad A \approx 0.25\text{cm}^2 = 0.25\times10^{-4}\,\text{m}^2$ in acid with $\rho = 1.2\times10^3\,\text{kg.m}^{-3}$
Then $\omega = 5.4\,\text{rads.s}^{-1}$ and $T = 1.16\,\text{s}$. Check this.

Example 2) A ship

A is deck area of ship



If the bottom area of the ship is roughly A and the draft is h, then the displacement mass equals $Ah\rho$.
The buoyancy force must balance the ship's weight, so

$$Ah\rho g = mg \quad \text{or} \quad m = Ah\rho$$

Therefore
$$\boxed{\omega^2 = \frac{g\rho A}{Ah\rho} = \frac{g}{h}}$$

So a ship bobs like a pendulum of length h.

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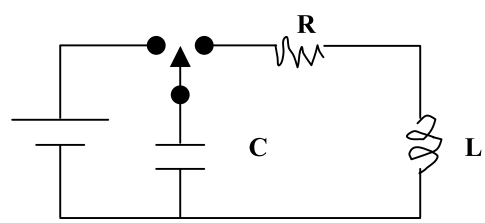
(5) The LC circuit

We can set up a current *i(t)* in an *LC* circuit by first charging up the capacitor *C,* and then discharging it through the inductance, *L*.

R ohms is a small resistance.
i(t) is the current flowing in circuit, and
Q(t) is the charge on the capacitance.



The induced voltage across the inductance is equal to that across the capacitance in order to keep the current low, ignoring the small voltage drop across R.

As the current is decreasing with time, $V_L = -L\frac{di}{dt}$, and $V_C = \frac{Q}{C}$.

Hence $-L\frac{di}{dt} = \frac{Q}{C}$.

So $\quad L\frac{di}{dt} + \frac{Q}{C} = 0 \qquad$ But $\; i = \frac{dQ}{dt}$

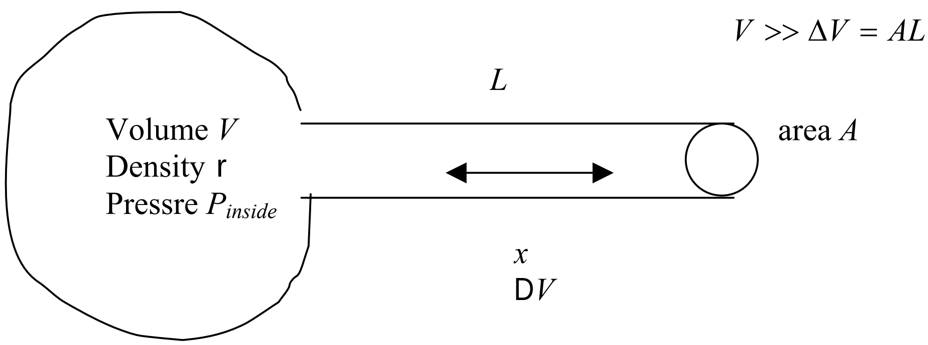
Hence $\frac{d^2 i}{dt^2} = -\frac{1}{LC} i = -\omega^2 i$

This is again a SHM equation with $\omega^2 = \frac{1}{LC}$.

The general solution is $i(t) = A\cos(\omega t + \varphi)$.

(6) Making sounds with a flask. The Helmholtz Resonator

Blowing across the open end of a bottle or flask produces a change in pressure at the opening. As an approximation we consider the neck of the vessel as containing a plug of air. This air moves inwards, and then *springs* back, due to the increased pressure Δp in the cavity over that outside, producing an outwards restoring force. The cavity acts like an air cushion or spring. This initiates SHM at a certain frequency, which we can then hear.



We need to relate the pressure difference Δp to the distance moved by the plug of air.

An important question is whether the spring constant of an air cavity is related to the isothermal or the adiabatic elasticity. The more realistic version is that of adiabatic elasticity.

When a gas is suddenly compressed it becomes warmer as a result of the work done on it; in other words, the particles composing it are moving faster, on the average. Since, according to the kinetic theory of gases, the pressure is proportional to the mean-squared molecular speed, this heating results in a greater restoring force than we would otherwise have, and the elastic modulus of the gas is larger than it would be under Boyle's law. Experience bears out this conclusion. Under completely adiabatic conditions (no flow of heat into or out of the gas) the pressure-volume relationship turns out to be

$$pV^{\gamma} = \text{constant}$$

From this we have $Ln(p) + \gamma Ln(V) =$ const

$$\frac{1}{p}\frac{\Delta p}{\Delta V} + \frac{\gamma}{V} = 0 \qquad \text{so} \qquad \Delta p = -\gamma p \frac{\Delta V}{V}$$

The appropriate elasticity modulus is that corresponding to changes in the total volume of the specimen associated with a uniform stress in the form of a pressure change over its whole surface. This is the bulk modulus K_{adiab}, defined in general as

$$\Delta p = -K_{adiab}\frac{\Delta V}{V}. \qquad \text{So} \qquad K_{adiab} = \gamma p.$$

The negative sign represents the fact that when we increase the pressure, the volume of gas decreases

The value of the constant γ is close to 1.67 for monatomic gases, 1.40 for diatomic gases, and is less than 1.40 for all others (at normal room temperatures). This enhanced elasticity under adiabatic conditions then increases the frequency of any vibrations involving enclosed volumes of gas. For a gas at atmospheric pressure this modulus is about five or six orders of magnitude less than for solid materials.

The equation of motion of the gas plug in the neck is

$m\ddot{x} = (\rho AL)\ddot{x} = F_x = -A(p_{inside} - p_{outside}) = -A\Delta p$, where Δp is the pressure difference.

But as $\Delta V = -Ax$ (a +ve x represents a decrease in volume). ρAL is mass moved

Therefore

$$(\rho AL)\ddot{x} = -A\Delta p = -A\left(K_{adiab}\frac{Ax}{V}\right)$$

$$\ddot{x} = -\left(\frac{K_{adiab}A}{\rho LV}\right)x = -\omega^2 x \quad \text{where} \quad \omega^2 = \frac{K_{adiab}A}{\rho LV}$$

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Orders of magnitude

For a gas of mass m, $\rho = \frac{m}{V}$ and $K_{adiab} = \gamma p$ for an ideal gas.

Hence

$$\boxed{\omega = \sqrt{\frac{\gamma Ap}{Lm}}}$$

For an ideal gas law for n moles we have $pV = nRT$.

Therefore $\frac{p}{m} = \frac{n}{m}\frac{RT}{V} = \frac{1}{M}\frac{RT}{V}$, where M is the mass of 1 mole.

$$\omega = \sqrt{\frac{A}{LV}\frac{\gamma RT}{M}}.$$

Now substitute a few values.

$V = 10^{-3}\,\text{m}^3$, $A = 10^{-4}\,\text{m}^2$, $L = 5\times10^{-2}\,\text{m}$,

air at STP $\gamma = 1.4$, $R = 8.3$ J.mole^{-1}K^{-1}, $M = 0.029$ kg.mole^{-1}, $T = 300$K

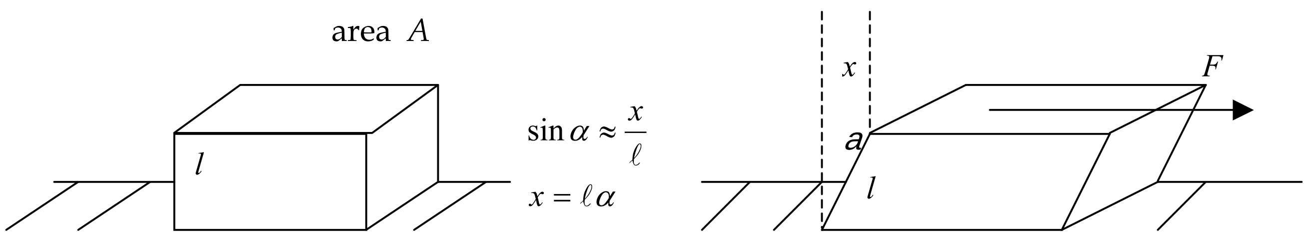
This gives $\omega = 490\text{s}^{-1}$ and $f = \frac{\omega}{2\pi} = 80\text{Hz}.$

This is 2 octaves below middle C.

7) The Torsion Pendulum

Although solid bodies are nearly rigid, they are not entirely rigid, and will deform if a large enough force is applied. A solid bar will extend or compress under large forces, but if the force is not too large the deformation is proportional to the force. In general the effect is elastic, returning to the original equilibrium position when the force is released. They obey Hooke's Law.

If one side of the body is clamped and the force pushes tangentially along the other side, then the deformation is called **shear**. During this deformation, the parallel layers of the body slide past each other, just like the pages of a book if you push on the cover.



before shear applied

after shearing

The shear modulus is defined as

$$\boxed{n = -\frac{F/A}{\alpha}}$$

Note $\frac{F}{A}$ is proportional to angle of deformation,

where A is the area over which the force F is applied. Note the negative sign as the deformation tries to return to the equilibrium position.

Typically n is equal to $(3-8)\times 10^{10}$ N.m^{-2}.

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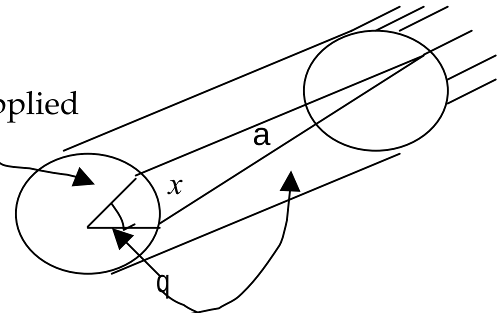
Now apply this to a cylindrical wire.
From the diagram we see that



$$x = r\theta = \alpha\ell \quad \text{so} \quad \alpha = \frac{r\theta}{\ell}.$$

Also

$$F = -n\alpha A = -\left(\frac{nAr}{\ell}\right)\theta \quad \text{and} \quad \text{Torque} = M = rF = -\left(\frac{nAr^2}{\ell}\right)\theta$$

if F is applied over an area A on the circumference

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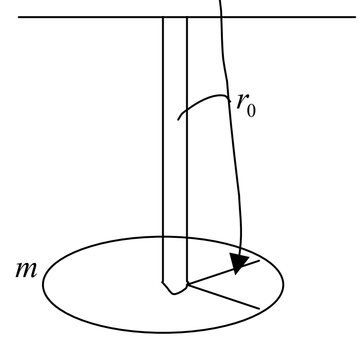
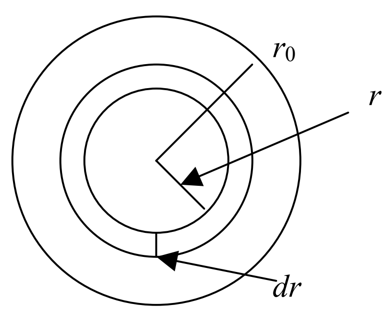
Apply the above to the Torsion Pendulum

The torsion pendulum consists of a solid cylinder of wire, radius r_0, attached to a disc of mass m, and radius R. The moment of inertia of the disc is $I = \frac{1}{2}mR^2$.



wire end-on face welded to disc.

Assume the wire's bottom face is welded to the disc. Consider element indicated
The torque is applied across a section of area $dA = 2\pi r dr$
is given by $dM = -\left(\frac{ndAr^2}{\ell}\right)\theta$



$$dM = -\left(\frac{n}{\ell}\theta\right)2\pi r^3 dr.$$

Therefore $$M = \int_{over\ face} dM = -\left(\frac{n}{\ell}\theta\right)2\pi\int_0^{r_0} r^3 dr = -\left(\frac{n\pi r_0^4}{2\ell}\right)\theta = -s\theta.$$

Then $I\ddot{\theta} = -s\theta$ giving SHM, with $\omega = \sqrt{\frac{s}{I}} = \frac{r_0^2}{R}\sqrt{\frac{n\pi}{\ell m}}$.

8) Vibrations of molecules

When two atoms are separated by a few atomic diameters, they exert an attractive force on each other. But if closer the Van der Waal's forces become repulsive. Between these limits there is an equilibrium position of the two atoms in the molecule. Disturbed from equilibrium, the atoms will oscillate with approximately simple harmonic motion.