FIELDS AND WAVES

5.0 OSCILLATIONS

Many physical systems undergo oscillatory motion, that is to say they repeat the same motion over and over again. Examples include the swing of a pendulum, or the vibration of a violin string; the oscillation of an electrical circuit or the beating of your heart. To help in the study of such motion, we will consider for the most part simple examples, which nevertheless are sufficiently general to introduce some of the key concepts which are useful also in more complicated situations.

The first characteristic property of an oscillating system is the *time period*, T, (measured in seconds) the time taken for the system to go through one complete *cycle* of its motion. And directly related to this is the *frequency*, f, which is the number of complete cycles completed in a second. The unit for f is the hertz (Hz).

$$f = 1/T.$$

So for example, if your heart beats 70 times a minute, its frequency is 70/60 Hz. The frequency of the electricity supply is 50 Hz, which means that every 1/50 th of a second the voltage varies through one complete cycle from 240V to -240V and back again.

5.1 SIMPLE HARMONIC MOTION

A particularly simple kind of oscillatory motion is *simple harmonic motion*, often abbreviated s.h.m. This is in many cases an excellent approximation to the full treatment even of realistic and more complicated systems. What is often the case is that the system oscillates about some *equilibrium* situation, and its behaviour can be described by the way that a variable called the *displacement* oscillates about the equilibrium value (which is taken to be zero). In a s.h.m., the displacement changes *sinusoidally* with time. So for example, the to-and-fro swing of a pendulum can be described by the oscillating angle the pendulum makes with the vertical. The equilibrium position is when the pendulum hangs straight down, and then the angle is zero. We can choose this angle to be the displacement, and it changes back-and-forth in value.

If the swing of the pendulum is small, and if we can ignore friction and other sources of *damping* which will eventually bring the pendulum to rest, it can be shown that the displacement does indeed vary in a sinusoidal fashion. That is to say, a graph of s against time t would be like a sine wave. We would be able to write

$$s = r \sin \theta$$

where θ is related to the time t by

$$\theta = \omega t.$$

Of course, the time we choose to call $t = 0$ is quite arbitrary, so a more general formula would be

$$s = r \sin(\omega t + \phi),$$

where ϕ is some constant. Please do not confuse the displacement s (which in this example is an *angle*, namely the angle made by the pendulum with the vertical) with the angle θ! This equation, $s = r \sin(\omega t + \phi)$, holds generally for the time dependence characteristic of a s.h.m.

You will see that since the function $\sin \theta$ repeats itself whenever the angle θ advances through $2\pi, \sin \theta = \sin(\theta + 2\pi)$, our formula for the displacement $s = r \sin(\omega t + \phi)$ does have the characteristic periodicity of an oscillatory motion, and furthermore that the time period T is such that

$$\omega T = 2\pi.$$

The quantity r is called the *amplitude* of the oscillation, and $\omega = 2\pi/T = 2\pi f$ is called the *angular frequency*; the unit for the angular frequency is rad.s^{-1}.

Since the displacement s is ever changing, we may also wish to consider at any instant how fast it is changing. This is given by $v = \mathrm{d}s/\mathrm{d}t$, or

$$v = r\omega\cos(\omega t + \phi).$$

It follows that v also undergoes s.h.m., and the same is true for *its* rate of change, $a = \mathrm{d}v/\mathrm{d}t$,

$$a = -r\omega^2\sin(\omega t + \phi).$$

You will observe that we have found

$$a = -\omega^2 s,$$

and this is in fact the defining characteristic property of a s.h.m.

5.2 OSCILLATIONS OF A LOADED SPRING

Before returning to the simple pendulum for a more complete description, let us consider another simple system which executes s.h.m. – a loaded spring. As was found by Robert Hooke, the tension in a spring is proportional to its extension. [Anxious to secure priority for his discovery, but at the same time not wishing that others might benefit from it before he had explored its consequences more fully, he published it in 1676 in the form of an anagram - *CEIINOSSITTUU* - which he only revealed in 1679 to be deciphered as *ut tensio, sic uis* meaning *as the extension, so the force*]. This is *Hooke's Law.* What it means is that when a spring is stretched, there is a *restoring force* which grows linearly with the extent to which it has been stretched; and conversely, if it is compressed, there is again a restoring force, again proportional to the extent to which it has been compressed. So if the length of the spring when it is neither stretched nor compressed is L , the tension T in the spring when it is stretched to the length $L + x$ is given by

$$T = kx,$$

where k is a constant called the *spring constant.* If the spring is compressed, the same equation continues to be valid; a negative value of x denotes the extent of the compression, and the negative value of T is again the restoring force acting to oppose the compression.

Hooke's law is an example of a phenomenological law. It is valid to good approximation so long as the extension is not too great. In fact there are many other situations in physics where a linear relationship, such as that between T and x holds for small values of the variables.

If a spring supports a mass M hanging from a beam, the tension in the spring acts upwards on the mass. There is also another force on the mass, namely its weight Mg which acts downwards. So if the unstretched length of the spring is L, when the mass is in equilibrium, with the tension in the spring exactly balancing the weight of the mass, the spring must have been stretched by an amount x_0 with

$$kx_0 = Mg.$$

If now the mass is displaced vertically from its equilibrium position by a displacement s (positive if downwards, negative if upwards) so that the length of the spring becomes $L + x = L + x_0 + s$, we have a tension $T = kx$ and the net downwards force on the mass is

$$F = Mg - T = Mg - kx = -ks.$$

Newton's law then shows that there is in consequence an acceleration a of the mass given by

$$a = -(k/M)s = -\omega^2 s.$$

We have written ω^2 for the ratio k/M. You may recognise that we now have the characteristic equation of a s.h.m. Note that the net force Fon the mass, and so also its acceleration, is in the opposite direction to the displacement. If the mass is displaced downwards from equilibrium, the net force pulls it back upwards, and *vice versa*, if the displacement is upwards, the net force is downwards.

The mass on the spring executes s.h.m. with angular frequency ω , so that we may write

$$s = r\cos\omega t.$$

[We have chosen the constants r and ϕ so that the displacement at time $t = 0$ is r, and the mass M has zero velocity at that time.] The time period T of the oscillations is given by

$$T = 2\pi\sqrt{(}M/k).$$

[Don't be confused by the use of the same symbol T here for the time period, and previously for the tension in the spring!]

5.3 THE SIMPLE PENDULUM

To return to the simple pendulum, consider a mass m (the bob on the pendulum) hanging on a string of negligible mass and fixed length L. When the string is displaced sideways, so that it makes an angle α with the vertical, the forces on the mass are the tension T in the string, and the weight mg. These forces are in different directions; what is now required is that we consider separately the forces and the accelerations they produce in two different directions [since the motion is confined to a vertical plane, this is sufficient]. It is convenient to choose these directions to be (a) radial, that is along the string; and (b) tangential, that is along the direction of the motion of the mass m in the sense of increasing angle α .

Since the bob moves on a circular path, there is an inwards acceleration, an acceleration in the radial direction, provided by the difference between the tension in the string (T) and the component of the weight force in that direction ($mg\cos\alpha$). The inwards (centripetal) acceleration is v^2/L. So the radial equation of motion gives

$$T - mg\cos\alpha = mv^2/L.$$

In fact we do not need to use this equation, unless we want to determine the tension in the string. What we do need is the equation obtained by considering the tangential direction. Here the tension of the string does not contribute, and the component of the weight, $mg\sin\alpha$ in magnitude, is in the opposite direction to that of increasing α. It produces an acceleration a of the mass m in the opposite direction to that of increasing α,

$$a = -g \sin \alpha$$

Now for small angles α we may approximate $\sin \alpha$ by s/L, where s is the linear displacement of the bob. This means that we have found

$$a = -gs/L = -\omega^2 s,$$

which is again the equation for s.h.m., with $\omega^2 = g/L$. The pendulum oscillates with s.h.m. with time period given by

$$T = 2\pi \sqrt{(L/g)},$$

and the motion is given (with a suitable choice of the constants α_0 and ϕ determined from the initial conditions) by

$$\alpha = \alpha_0 \sin(\omega t + \phi),$$

which on multiplying through by L allows one to write (for small α_0)

$$s = r \sin(\omega t + \phi).$$

5.4 ENERGY CONSIDERATIONS

The mass on a spring, or the bob on the pendulum, has a kinetic energy which varies through the cycle of motion, and so also does the potential energy. For the mass on the spring, this is the energy stored in the spring when it is extended or compressed. For the pendulum bob, this is the gravitational potential energy associated with its rise above the equilibrium level as the pendulum swings from side to side.

The kinetic energy of the mass on the spring is just $Mv^2/2$, where the speed v is given by

$$v = -r\omega \sin \omega t;$$

$$K.E. = Mr^2\omega^2 \sin^2 \omega t/2.$$

The potential energy stored in the spring as it is stretched or compressed by the displacement s from its length at equilibrium is given by

$$P.E. = ks^2/2 = Mr^2\omega^2 \cos^2 \omega t/2.$$

The K.E. and the P.E. both oscillate (with frequency twice that of the displacement), but their sum stays constant; $K.E. + P.E. = Mr^2\omega^2/2$.

Similar statements can be made for the pendulum. The speed of the bob is $v = \omega rcos(\omega t + \phi)$, and its kinetic energy is

$$K.E. = mv^2/2 = m\omega^2 r^2 \cos^2(\omega t + \phi)/2.$$

Likewise, when the string is at an angle α with the vertical, the bob is at a height $L(1-\cos\alpha)$ above its equilibrium position, so having gained in potential energy an amount

$$P.E. = mgL(1-\cos\alpha) \approx mgL\alpha^2/2 = mgL\alpha_0^2 \sin^2(\omega t+\phi)/2 = m\omega^2 r^2 \sin^2(\omega t+\phi)/2.$$

5.5 DAMPING

So far we have ignored the consequence of friction, air resistance, and the like which in a realistic situation will lead to the amplitude of the oscillations getting smaller and smaller. This is called *damping* of the oscillations. A full treatment of damping is beyond the bounds of this course, but in qualitative terms we can recognise that there is a time characteristic of the way that the amplitude diminishes – a long time if the system is *lightly damped* and a shorter time if it is *heavily damped*. In a system with light damping there are still oscillations; the displacement changes sign again and again as it goes through equilibrium, but makes ever decreasing excursions away from equilibrium. With heavy damping, the displacement falls away to zero without changing sign. Between these two extremes there is what is called *critical damping* which is when the friction or similar dissipative forces are just such as to bring the displacement back to zero as quickly as possible without any overshoot.

5.6 RESONANCE

A system with both the restoring force characteristic of s.h.m. and damping can be *driven* by some further force. So a child on a swing (like a pendulum) can be pushed regularly and made to swing too and fro in a regular fashion. The amplitude of the resulting oscillation will depend not only on the force applied with each push, but also on the frequency with which the pushes are applied. When this frequency is just right, the resultant amplitude of the swing becomes greatest; this is an example of what is called *resonance*. Other examples of resonance are familiar from acoustics; a string on a guitar or a violin can be made to vibrate by sounding just the right note nearby. Or a poorly designed loudspeaker will resonate at certain frequencies. The resonant frequency is close to the natural (undamped) frequency of the system concerned.

Notes prepared by John M Charap 09/02/04