FIELDS AND WAVES

0.1 INTRODUCTION

This course continues the study of "classical" Newtonian physics you will have started last semester. The first four areas of study to be covered deal with fields. Fields occupy a prominent place in all of present-day physics, so this topic is of central importance. The gravitational and electromagnetic fields you will meet in the next few weeks are still discussed in contemporary research. Of course, the questions now being addressed are different from those we will be treating in this course. But the ideas we will introduce in these lectures are nevertheless relevant to physics today.

Later in the course, the focus will shift to waves. Waves, and oscillatory phenomena are likewise of topical relevance both in research and in practical applications of physics. Indeed it is hard to imagine how any understanding of physics can proceed without encountering waves and related phenomena. You may have heard of "wave-mechanics", which is one way to formulate quantum mechanics. So what you will study about waves in this course has application also in understanding quantum mechanics and so also most of present day advances in physics.

1.1 GRAVITATIONAL FIELDS

Let me start by reminding you of Newton's three laws of motion:

• Every object continues at rest or with uniform velocity unless acted upon by a resultant force.

• The rate of change of momentum of an object is proportional to the resultant force which acts on the object.

• When two objects interact, they exert equal and opposite forces on one another.

These laws were sufficient to account for all the known phenomena involving such interactions as arise in everyday life; but one of the most familiar of all situations presented a problem. Falling objects accelerate, and as was well understood ever since the careful and clear account made by Galileo, this is because they are acted on by a force - gravity. But unlike the forces exerted by a horse pulling a cart, or the friction which resists its motion, the gravitational force on a body, its weight, seemed not to require a direct contact between the body and anything else. It was clear that objects close to the surface of the Earth fell towards it, and so it seems as though there is a force exerted by the Earth which attracts the body, and so is responsible for its weight. Furthermore, it was apparent that this downwards force is in fact directed towards the centre of the Earth. But it acted at a distance.

As you know, with brilliant insight Newton proposed a Universal Law of Gravity in which for the first time the laws of mechanics were extended from applying to things terrestrial to include also under the same universal laws the motions of the planets and the Moon. Expressed as an equation, Newton's law of gravitation (for objects treated as points) is:

$$F = -G\frac{m_1 m_2}{r^2},$$

where F is the gravitational force between two objects (with masses m_1, m_2), distance r apart. The minus sign indicating that the force is always in the opposite direction to increasing r, which is to say that it is always attractive. The constant G, is the universal constant of gravitation, one of the fundamental constants of Nature. Its value is

$$G = 6.673 \times 10^{-11} \mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{kg}^{-2}.$$

This was first measured by Maskelyne in 1774 (as you will see in the video), but the first reasonably accurate measurement was made in 1798 by Henry Cavendish. It is very difficult to measure precisely, and it is the least well-determined of all the fundamental constants. Our way of expressing the law of gravitation, and at the same time removing the difficulty with action-at-a-distance is by introducing the concept of the gravitational field. Space is no longer to be considered as empty, but rather as occupied by fields, with the gravitational field as the first example we will consider. The gravitational force exerted by one mass on another is ascribed to the gravitational field. The mass of an object produces a force field which extends throughout space, and it is this force field which then acts on the mass of each and every other object, attracting it to the first.

The gravitational field produced by a mass M is proportional to M. And its effect on a mass m is proportional to m. This has the immediate and important consequence that the acceleration that the force on m produces is independent on m. This explains the famous (but probably apocryphal) experiment by Galileo at the Leaning Tower of Pisa. It is also one of the pillars on which centuries later Einstein was to build the General Theory of Relativity. The gravitational field at any point P is defined to be the gravitational force that would be experienced by a small "test mass" m placed at P, divided by m. It is therefore just the acceleration which that test mass would experience. We will write the magnitude of the gravitational field as g. Note that the units in which g is to be expressed are those of force divided by mass, so $\text{N}\cdot\text{kg}^{-1}$ (and this is in fact the same as the more familiar units $\text{m}\cdot\text{s}^{-2}$ for acceleration). Since it has a direction (after all, it is proportional to a force) the gravitational field is in fact a vector, and if we wish to emphasise this, we should write it as $\mathbf{g}$. [It is a very special property of the gravitational field that the strength g of the field is equal to the acceleration it produces.] The strength of the gravitational field produced by a particle of mass M at distance r is easily seen to be

$$g = \frac{F}{m} = \frac{GM}{r^2}.$$

This same equation is valid even if the object of mass M is not point-like, so long as is spherically symmetrical, and so long as the distance r is greater than its radius. So even outside a planet or a star (which we can suppose to be spherically symmetrical) the same inverse square law holds. In order to derive this result, Newton had first to invent integral calculus, which may explain why he waited so long (from 1665 or 1666 when, according to what he wrote fifty years later, he had first conceived his universal law) to 1687 when he published it in his masterpiece, the *Principia*. (What he showed was that the gravitational field outside of a thin uniform spherical shell of matter was the same as though all of its mass was concentrated at its centre; and inside such a shell, its gravitational field was zero). The equation above may be used to obtain the surface gravity of a planet or star (or of the Moon), by setting r equal to the radius of the planet or star (or the Moon).

As with all vectors, it is often helpful to represent the gravitational field at a point by an arrow, with the direction of the field specifying the direction of the arrow, and its length indicating its magnitude. This would give us a picture of the gravitational field something like the picture of the wind velocity you will have seen in the weather report and forecast. [The wind velocity is indeed a vector field!] But it is sometimes more helpful to trace out what are called the field lines, in which through any point P there is a line, with the tangent to the line at P giving the direction of the field at that point. If the field lines are straight, the field is said to be parallel, and if furthermore the magnitude of the field is constant, the field is said to be uniform. The gravitational field in a small enough region close to the Earth is often approximated by taking it as uniform; its direction is then vertically downwards and its magnitude is — well, what is familiarly called g, with the value 9.8 $\mathrm{N{\cdot}kg^{-1}}$, more usually quoted as 9.8 $\mathrm{m{\cdot}s^{-2}}$. A non-uniform field does not have the same magnitude and direction at all points. We may anticipate that the gravitational field of the Earth is in fact non-uniform, although in a small enough region it may be treated as uniform. "Small enough" has to be interpreted in a sensible way. The departure from uniformity may safely be ignored when dealing with, say, a game of football; but not for sending a rocket to the Moon.

1.2 THE GRAVITATIONAL POTENTIAL

The gravitational field due to a number of different masses may be obtained by adding together the fields produced by each of them separately; likewise for a distribution of masses, the resultant field may be obtained by integration. But adding vectors is not so easy as adding scalar quantities! So it is very useful to be able to describe the gravitational field by introducing a scalar quantity - the gravitational potential. This is closely related to the gravitational potential energy. You will know that as an object moves in the direction of the gravitational force (for example by falling!), it can be made to do work. Conversely, to lift it against the gravitational force (its weight!) requires that work should be done, or energy expended. The gravitational potential energy PE(A) of a test mass at rest at the point A is defined to be minus the amount of work needed to take it "to infinity", so far away that the gravitational field is negligible. Note that it is always negative. To move it from A to B then requires work equal to PE(B) - PE(A) (which can be positive or negative, depending on whether the gravitational force is hindering or helping). If we write PE(r) for the potential energy of a test particle of mass m at distance r from a planet (or any spherically symmetric object) of mass M, it is then clear that the work needed to move it from r to s is PE(s) - PE(r). In particular, to move it from r to $r+\Delta r$ requires PE($r+\Delta r$)- PE(r); but this is $\Delta W = -F(r) \cdot \Delta r$ (the minus sign because this is the work done *against* the force, not the work done *by* the force), where as we have already seen $F(r)$ is

$$F(r) = -G\frac{mM}{r^2},$$

so we may write

$$\Delta\text{PE}(r) = \text{PE}(r + \Delta r) - \text{PE}(r) = G\frac{mM}{r^2} \cdot \Delta r.$$

mM The gravitational potential V is defined as the gravitational potential energy of the test mass divided by its mass, so we have

$$\Delta V(r) = V(r + \Delta r) - V(r) = G\frac{mM}{r^2} \cdot \Delta r.$$

This equation may be integrated so as to give

$$V(r) = -G\frac{M}{r}$$

as the gravitational potential energy at distance r outside the planet. This is minus the work needed per unit mass to take an object from r to infinity; it is negative, since it always takes work to lift the object against the force of gravity to remove it to infinity. Since work is measured in joules, the units for V are J.kg^{-1} (since J = kg.m^2.s^{-2}, this is also - but less conventionally - the unit for the square of a velocity).

Note that the potential energy at a height h above the surface of the Earth (radius R) is then given by putting $r = R + h$. If h is reasonably small compared with the radius of the Earth (R), it is then possible (by using the binomial theorem) to approximate the resulting expression by first setting

$$(R + h)^{-1} = R^{-1}(1 + h/R)^{-1} \approx R^{-1}(1 - h/R),$$

and then since $h << R$, we have

$$V(R + h) - V(R) \approx \frac{GM}{R^2}h = gh,$$

where we have recognised that the usual definition of the work done in raising a unit mass from the surface of the Earth to a height h is just gh, where now

g is the familiar acceleration due to gravity, 9.8 m s^{-2}. We can learn two useful things from this. First, there is the relation

$$g = \frac{GM}{R^2}$$

giving the acceleration due to gravity near the surface of the Earth in terms of G, and the mass M and radius R of the Earth; of course exactly similar arguments will allow us to calculate the surface gravity (as it is called) for the Moon, a planet, the Sun or a star.

To check this result, note that for the Earth, $M \approx 6.0 \times 10^{24}$ kg, $R \approx 6.4 \times 10^6$ m, and with $G \approx 6.7 \times 10^{-11}$ N m^2 kg^{-2}, we find

$$g \approx (6.7 \times 10^{-11}\ \mathrm{Nm^2kg^{-2}}) \times (6.0 \times 10^{24}\mathrm{kg}) \times (6.4 \times 10^6\mathrm{m}) \approx 9.8\mathrm{Nkg^{-1}}$$

as we should expect.

Of course we could also use the equation for g together with the radius of the Earth, which had already been determined in antiquity, to deduce from Cavendish's measurement of G what is the value of M, the mass of the Earth! The other thing we can note is the generalisation of this result. You can see that what we have found is that

$$g = -\frac{\mathrm{d}V}{\mathrm{d}r};$$

the minus sign here is to indicate that the gravitational field is down, in the opposite direction to that of increasing r. And this result is quite general. The gravitational field is minus the gradient of the gravitational potential.

A useful aid to thinking about the potential is to consider equipotentials. These are like contour lines on a map, which are lines of constant elevation above sea-level, or isobars, lines of constant barometric pressure. An important difference is that instead of lines, we now have equipotential surfaces, although we will often draw them as lines (a section through the surface). So the equipotentials around a planet, the Earth for example, will be concentric spheres, which we can draw as circles. If the equipotentials (for equal potential differences) are close together, the field strength is large, and if they are far apart it is weak - just as the gradient of a hill is large where the contour lines (for equal vertical separation) are close together and small where they are far apart. Also the direction of the field is normal (perpendicular) to the equipotentials; the field lines cut the equipotential surfaces at right angles, just as the paths of steepest descent cross the contour lines at right angles (which is why rivers and streams cross the contour lines on a map at right angles).

1.3 KEPLER'S LAWS AND SATELLITE MOTION

Tycho Brahe (1546-1601), a Danish nobleman, built an observatory on the island of Hveen from which over a period of some twenty years he made meticulous observations of the heavens, and in particular of the motions of the planets. In 1599 he was appointed Imperial Mathematician to the Holy Roman Emperor, King Rudolph II in Prague. He was joined there by Johannes Kepler (1571-1630) as his assistant. Kepler deduced from Tycho's data that the planets followed what we now know as Kepler's Laws. These are:

1. Each planet moves in an elliptical orbit with the Sun at one of the two focal points of the ellipse.
2. The line from the Sun to each planet sweeps out equal areas in equal times.
3. The ratio a^3/T^2 is the same for all the planets.

[T is the period, the time taken for the planet to make one complete orbit, and a is one half of the longest cord of the ellipse.] Newton was able to show that all three laws follow from his universal law of gravity. The same laws are obeyed also by satellites in orbit around a planet, but with a different constant in the third law. In fact the constant is $GM/4\pi^2$.

1.4 ESCAPE VELOCITY AND BLACK HOLES

In order to send a rocket up into space, work must be done against the gravitational field. And we can see that this has to exceed the difference between its gravitational potential energy at the height to which we wish to send the rocket and the gravitational potential energy at the launch: W = PE(max height) - PE(launch). To make the rocket escape completely (”to infinity”) we need $W \geq$-PE(launch), and this might for example be provided by giving it a vertical boost into space with an initial velocity v(launch). The work done as it climbs into space then comes from its initial kinetic energy = $mv(\text{launch})^2/2$. So to launch it with a velocity great enough for it to escape altogether requires $mv(\text{launch})^2/2 \geq$ - PE(launch). But the right hand side of this equation is just GMm/R. The factors m cancel out, and we find for the escape velocity (the minimum launch velocity required to escape to infinity)

$$v(\text{escape})^2 = 2GM/R,$$

where M and R are respectively the mass and the radius of the Earth. This gives $v(\text{escape}) = 11 \times 10^3$ m s^{-1} as the escape velocity from Earth. Similar calculations can be done for other astronomical bodies. Laplace speculated about what might happen if the escape velocity equalled c, the speed of light. The formula would then give $R_c = 2GM/c^2$. Although Laplace's arguments are not really appropriate, since we should really use the ideas of general relativity for such a discussion, the result obtained for the radius R_c is in fact correct. It is called the Schwarzschild radius of the mass M, and if the mass all resides within the Schwarzschild radius there indeed results an object from which neither light, nor indeed any material object which perforce has a speed less than that of light, can escape. It is a black hole. You might ask whether such things exist. We have very good evidence that they do. There are stars which have exhausted their thermonuclear source of energy, and then collapsed under their own weight; they reveal themselves as powerful sources of x-rays generated by matter falling into them. There are also supermassive black holes, millions of times more massive than the Sun. These have been identified at the centre of some galaxies, and it is highly probable that there may be such a black hole at the centre even of our own Galaxy.

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