Wrg-12 Superposition of Waves

You need to know the following trigonometric identity.

$$\cos A + \cos B = 2\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

We wish to find the resultant of the superposition of two waves (with equal amplitudes and reasonably close parameters)

Just as for vibrations we can evaluate $y = y_1 + y_2$ using the above trig identity, with

$A = k_1 x - \omega_1 t$ and $B = k_2 x - \omega_2 t$ Both waves moving to the right.

Hence

$$y = y_1 + y_2 = A\cos(k_1 x - \omega_1 t) + A\cos(k_2 x - \omega_2 t)$$

$$y = 2A\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\cos\left(\bar{k}x - \bar{\omega}t\right)$$

where

$$\Delta\omega = \omega_2 - \omega_1, \quad \bar{\omega} = \frac{\omega_1 + \omega_2}{2}$$

$$\Delta k = k_2 - k_1, \quad \bar{k} = \frac{k_1 + k_2}{2}$$

This method is easy for two equal amplitude waves, but not for other cases. A complex number method makes it possible to solve more complicated problems.

Beats, Wave Packets, Group Velocity

When one adds two infinite length wavetrains one obtains

$$y = 2A\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\cos\left(\bar{k}x - \bar{\omega}t\right)$$

The " amplitude" travels slowly with $\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t = \text{constant}$

So velocity $\equiv$ $v_g = \frac{dx}{dt} = \frac{\Delta\omega}{\Delta k} \to \frac{d\omega}{dk}$ and

wavelength $\equiv$ $\lambda_g = \frac{2\pi}{\left(\frac{1}{2}\Delta k\right)}$ …… (a)

Wave travelling at $v_p = \frac{\bar{\omega}}{\bar{k}}$ with $\lambda = \frac{2\pi}{\bar{k}}$

The Group or Pulse or Wave Packet moves with a velocity of v_g and a wave packet size of $\approx \Delta x$.

Bandwidth Theorem

The size of the packet is $\equiv \Delta x \approx \dfrac{\lambda_g}{2} = \dfrac{2\pi}{\Delta k}$ from equ (a).

Hence $\boxed{\Delta k \Delta x \approx 2\pi}$

Note that

$\Delta k = \dfrac{\Delta \omega}{v_g}$ measures the spread in frequencies (or wavelengths) making up the wave packet.

$\Delta x =$ the spatial extension of the wave packet, corresponding to the size of the wave packet.

The Bandwidth Theorem tells us

- that in order to obtain a sharp pulse (one well localised in space) we need to superimpose a large number and range of different frequencies.

The sharper and more isolated the pulse (ie the more localised), the more frequencies are needed.

Examples

(a) An infinitely long wave-train of one frequency (monochromatic). It is not localised at all, and the energy of the wave is spread over all space. This extreme case corresponds to $\Delta x = \infty \quad \Delta k = 0$

(b) A few frequencies added. Separated packets, but still an infinitely long train.

(c) A single isolated pulse. Composed of an infinite number of frequencies. Localised at a certain position. This extreme case corresponds to $\Delta x = 0 \quad \Delta k = \infty$.

In general:

v_g

______ Δx _______

The reason why we need an infinity of infinitesimally separated frequencies to produce an isolated wave packet is that the superposition $\sum_n C_n \cos(k_n x - \omega_n t)$ must achieve complete cancellation everywhere except over a small range Δx, for all time.

Practical Consequences

a) Light:

For light and other electromagnetic waves we can never produce a perfectly monochromatic wave train because the Bandwidth Theorem requires $\Delta x = \infty$ for $\Delta k = 0$,
and any source produces radiation by processes taking a finite time (atomic transitions), and not in step with each other (incoherent). Therefore $\Delta x \neq \infty$.

The best nearly monochromatic sources are MASERS and LASERS, which achieve coherence over long time periods by stimulated emission. The radiation itself provides the stimulations to make other atoms emit in step (ie in phase, or coherently) with the radiation.

b) An acoustic example:

Acoustic engineers test the acoustic properties of a concert hall by using a pistol! Why?
A gunshot is a very short pulse of sound, Δx is very small. Therefore $\Delta k = 2\pi / \Delta x$ is very large, so the gunshot contains very many frequencies. Hence we can simultaneously test the concert hall's response to a large range of different frequencies.

Essentially the gunshot produces a wave packet $y_{IN}(x, y)$
The engineer's equipment detects, measures and Fourier analyses the response of the Concert Hall.

c) Quantum Mechanics:

Here $y(x,t)$ is called the wave function and written as $\Psi(x,t)$.

Note the De Broglie relationship; momentum of a particle, $p = \frac{h}{\lambda} = \hbar \frac{2\pi}{\lambda} = \hbar k$.

Therefore $\boxed{\Delta p = \hbar \Delta k}$.

Since the wave function is a quantum mechanical wave, the Bandwidth Theorem holds, $\Delta k \Delta x = 2\pi$.

Therefore $\Delta p \Delta x = \hbar \Delta k \Delta x = 2\pi\hbar = h$, and so $\boxed{\Delta p \Delta x = h}$.

This is called the Heisenberg Uncertainty Principle. We have shown that this is a consequence of the wave nature of the wave function $\Psi(x,t)$.