REMEMBER

-The general form of the wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$; $v = \frac{\omega}{k}$. For waves light it becomes $\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$; **$c \approx 3*10^8$m/s** ; E is the electric field.

-As all waves need a medium to propagate, Maxwell proposed the "ether" as the medium that propagates the light. If existed, it would be the *absolute frame* for all laws of physics.

-Michelson and Morley used the earth revolution around the sun and the interferometer of Michelson to prove the presence of ether. They found a negative result.

-The mechanics' laws are covariant. The Maxwell theory was not covariant; it was right only for the absolute system of reference "the ether".

- **Special** theory of relativity mean "**only for inertial frames"**. The principle of relativity ***All physical laws have the same form in all inertial frames***. It requires "covariance" for all physical laws. Consequence: No need for the "ether" any more. To make this principle works, one has to use Lorentz transformations instead of Galilean transformations. Note that for low speed the Lorentz transformations fits with Galilean transformations.

- The second principle: ***The speed of light in vacuum is the same in all inertial frames of reference and is independent on the motion of the source or the observer.***
CONSEQUENCE: ***An inertial observer (or any material particle) cannot travel at speed c. The speed of light in vacuum is a maximum unattainable by any object.***

- A physical measurement is an *event* with coordinates (x, y, z, t) in the frame S. The same *event* has the coordinates (x', y', z', t') in the frame S'. A **rest frame** is the inertial frame in which the ***object is at rest***. **The proper length L_0 of an object is the space interval between its ends measured in object's rest frame**. The length of the same object measured in another frame moving with speed ***v*** **with respect to its rest** frame is

$$L = \frac{L_0}{\gamma} = L_0\sqrt{(1-\beta^2)} = L_0\sqrt{\left(1-\frac{v^2}{c^2}\right)}; \qquad \beta = \frac{v}{c}; \qquad \gamma = \frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}} = \frac{1}{\sqrt{(1-\beta^2)}}$$

Here are some values of γ ($1 \le \gamma < +\infty$) for different speeds ($0 \le \beta < 1$)

v / c = β	0.6	0.8	0.98	0.995	0.9965	0.9992
γ	5/4	5/3	5	10	12	25

-The interval of time between two events depends on the frame of time measuring. In general, two events that happen in different space locations and one need **two watches** to measure the time interval between them. But, there is one frame where only one watch is sufficient and the two events happen at the same location. **It is the rest frame of this watch; here the time interval between the two events is shortest T_0**. The time interval between the two same events at any other frame is longer (*time dilatation*) **$T = \gamma T_0$**

RELATIVISTIC DOPPLER EFFECT

-Consider a light source in front of a train moving with speed ***v*** toward an observer O at rest on an inertial frame S (fig.1). Another observer O' in the train (frame S') measures the wave frequency of light in train. He finds the value $\mathbf{f_0}$ and calculates the wave period

$$T_0 = \frac{1}{f_0} \quad (1)$$

$\mathbf{T_0}$ is the *interval of time* between emissions of two successive crests. Note that it is **the proper time** of the two corresponding events **(**crests appearing at the O' watch**).**

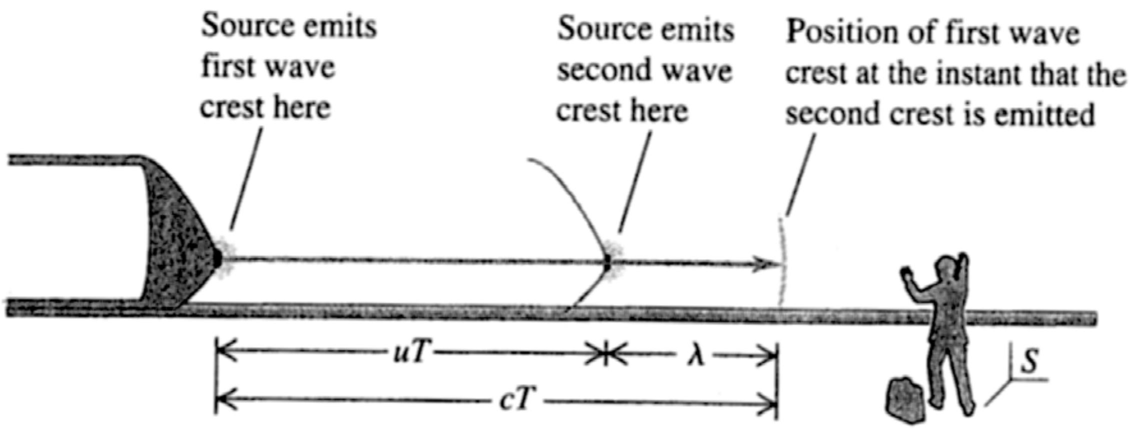


Fig 1

- Let **T** be the time interval one crest spends to arrive from source to the observer O in the frame S (we know it is different from $\mathbf{T_0}$). Taking into account that:
  a) The train emitted this crest when it was at the distance **c*T** from observer O;
  b) During this time, the train has advanced by ***v**T;**
  c) After time t the light source emits the following crest,

It comes out that when measuring the distance between two successive crests, that is the wavelength, the observer O finds (see fig 1)

$$\lambda = c*T - v*T = (c-v)*T \quad (2)$$

- Then the observer O calculates the wave frequency as

$$f = c/\lambda = \frac{c}{c-v}*\frac{1}{T} = \frac{1}{1-v/c}*\frac{1}{T} \quad (3)$$

As the observer "*is moving*" (velocity "***v***") with respect to the system S', the interval of time T between the two events can be calculated from the proper time of interval $\mathbf{T_0}$ as

$$T = \gamma * T_0 \quad (4)$$

One may find that

$$f = \frac{1}{1-v/c} * \frac{1}{T} = \frac{1}{1-v/c} * \frac{1}{\gamma * T_0} = \frac{1}{(1-v/c)} * (1-v^2/c^2)^{1/2} * f_0 =$$

$$= \frac{[(1-v/c)*(1+v/c)^{1/2}]}{(1-v/c)} * f_0 = \frac{(1+v/c)^{1/2}}{(1-v/c)^{1/2}} * f_0 = \sqrt{\frac{c+v}{c-v}} * f_0$$

So, the measured frequency in frame s is

$$f = \sqrt{\frac{c+v}{c-v}} * f_0 \qquad (5)$$

- Note that this expression contains only the relative velocity of source versus the observer. This happens because we do not have a transporting medium for light (**no ether**). In the classical Doppler effect one had to deal with two velocities **Vs** source speed and $\mathbf{V_O}$ observer speed because of the propagating medium "the air at rest".

- The formula (5) corresponds to the case when the source approaches the observer. One may easily find that when the source moves away from the observer

$$f = \sqrt{\frac{c-v}{c+v}} * f_0 \qquad (6)$$

As easily seen from (5) and (6) the observer O measures $f > f_0$ for approach and $f < f_0$ When the source and observer move far away to each other.

- For "normal" situations on the earth the velocity $\boldsymbol{v} \ll \boldsymbol{c}$ and one may find a simple formula for Doppler shift. Lets consider the expression (5)

$$f - f_0 = (\sqrt{\frac{c+v}{c-v}} - 1) * f_0 \rightarrow \frac{\Delta f}{f_0} = \frac{\dfrac{(c+v)^{1/2} - (c-v)^{1/2}}{(c-v)^{1/2}}}{\dfrac{(c+v)^{1/2}}{(c-v)^{1/2}}} = \frac{(c+v)^{1/2} - (c-v)^{1/2}}{(c+v)^{1/2}} =$$

$$= \frac{(1+v/c)^{1/2} - (1-v/c)^{1/2}}{(1+v/c)^{1/2}} \cong \frac{1 + \frac{1}{2} * \frac{v}{c} - 1 + \frac{1}{2} * \frac{v}{c}}{1 + \frac{1}{2} * \frac{v}{c}} = \frac{\frac{v}{c}}{1 + \frac{1}{2}\frac{v}{c}} \cong \frac{v}{c}$$

So, we get the expression

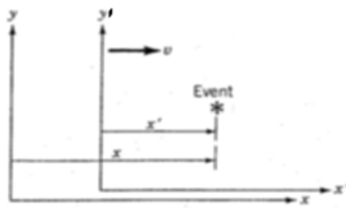
$$\frac{\Delta f}{f_0} = \frac{v}{c} \qquad (7)$$

which is very simple and practical for calculating the Doppler shift for all electromagnetic sources in movements. (Example; Police handheld radar) .

THE LORENTS TRANSFORMATIONS

-The Galilean transformations (frame S (x, t) / frame S'; x'=x + v*t and t' = t) do not accept the second principle of relativity. Also, the Maxwell theory is not covariant with respect to Galilean transformations. So, Einstein decided to use another type of transformations that accept the second principle and make function the covariance of Maxwell equations. These are the Lorentz transformations.

-Consider two inertial frames (fig. 2) S and S' (moving with speed ***v*** with respect to S along Ox axis). Any event has coordinates (x,y,z,t) in frame and (x',y',z',t') in frame S'. As there is no length changes for directions perpendicular to Ox, it comes out that O'y' = Oy and O'z' =Oz. Meanwhile, we know that there is change in lengths along Ox axis and change in interval time values. Lorentz transformations show that



$$x' = \gamma(x - v*t) \quad (8)$$

$$t' = \gamma(t - \frac{v}{c^2}*x) \quad (9)$$

Fig 2

- Equation (9) tells that the variable t' depends on t and x. This proves that Einstein definition for an event as a set (x,y,z,t) is essential. ***In the theory of relativity the time and the space merge together to form a space-time reality.*** In fact, when dealing with relativity issues, the scientists use the notations (x_1, x_2, x_3, x_4) for an event.

$$x_1 = x; \_\_\_ x_2 = y; \_\_\_ x_3 = z; \_\_\_ x_4 = c*t \quad (10)$$

By using these coordinates, the expression (8) and (9) transform to

$$x_1^{'} = \gamma[x_1 - v/c*(ct)] = \gamma[x_1 - \beta * x_4] \quad (11)$$

$$x_4^{'} = c*t^{'} = \gamma(c*t - \frac{v}{c}*x_1) = \gamma(x_4 - \beta * x_1) \quad (12)$$

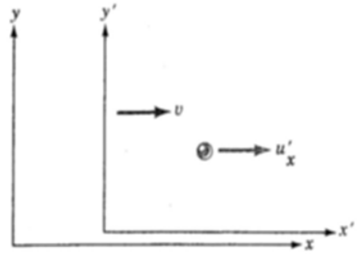
- The Lorentz transformations return to Galilean transformations for "normal" values of speed because for $v << c$ one has $v/c^2 = 0,\ \gamma = 1$ ***and*** $x' = x - v*t$; $t' = t$
Lorentz transformations are fully symmetrical. One can calculate the coordinates in system S by using the same expressions (8) (9). One has only to take into account that velocity must be negative; So.

$$x = \gamma(x' + v*t') \quad (8)$$

$$t = \gamma(t' + \frac{v}{c^2}*x') \quad (9)$$

THE ADDITION OF VELOCITIES

-Consider the inertial frame S' is moving with velocity ***v*** with respect to frame S and along the Ox axis (fig 3). One particle moves with velocity ***u'*** in frame S'. We know that, in classical mechanics, its corresponding velocity with respect to frame S is calculated on the basis of Galilean transformations (frame S (x, t) / frame S'(x',t') x = x' + v*t and t' = t). The velocity comes out as derivative



$$u = \frac{dx}{dt} = \frac{dx'}{dt} + v = u' + v \qquad (10)$$

Fig 3

- In relativity theory we do not have t' = t. So, we have to proceed following the formal algebra to find the expression for the derivative (u = dx/dt). Based on expressions 8-9

$$dx = \gamma(dx' + v * dt') = \gamma * dt'(\frac{dx'}{dt'} + v) = \gamma * dt'(u' + v) \qquad (11)$$

$$dt = \gamma(dt' + \frac{v}{c^2} * dx') = \gamma * dt'(1 + \frac{v}{c^2} * \frac{dx'}{dt'}) = \gamma * dt'(1 + \frac{v}{c^2} * u') \qquad (12)$$

$$u = \frac{dx}{dt} = \frac{\gamma * dt'(u' + v)}{\gamma * dt'(1 + \frac{v}{c^2} * u')} = \frac{u' + v}{1 + \frac{v}{c^2} * u'} \qquad (13)$$

-Note that:

a) The expression (13) transforms to classical way of velocities' addition when one deals with "normal" velocities because ***v, u' << c*** and [1+(v/c)*(u'/c)] = 1.
b) When the "particle" is an electromagnetic photon with ***u' = c*** , the expression (13) produces a result that respects the second principle of relativity

$$u = \frac{c + v}{1 + \frac{v}{c^2} * c} = \frac{c + v}{c + v} * c = c \qquad (14)$$

RELATIVISTIC DYNAMICS

- We showed that the Lorentz transformations accept the constancy of light speed. This is one requirement derived from the Maxwell theory. A more deep mathematical calculus proves that Maxwell theory is fully covariant with respect of Lorentz transformations. Now, it remains to make the necessary corrections in the classical mechanics so that it becomes covariant with respect to Lorentzian transformations, too. Here we proceed only with two major issues of mechanics; ***energy and linear momentum***.

- Consider one closed box with a light source S and a detector D at two ends (fig 4). The box ***is at rest in vacuum***. Let L be its length and M the total mass (box + S+ D).

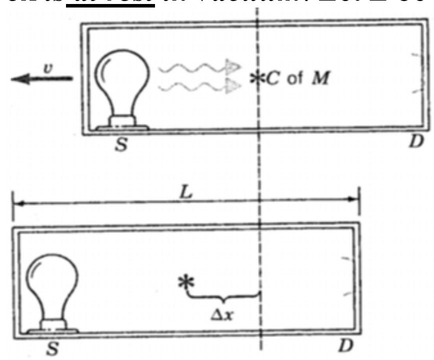


Fig 4

Suppose that the source emits a light pulse versus D. The light pulse is modelled by a wavelet. It transports its energy (~amplitude ^{2}) from S to D. Maxwell theory shows that this energy transport is associated by a transport of linear impulse

$$p = E / c \qquad (15)$$

E is the energy transported by light pulse.
p is the linear momentum transported by pulse.

b) The box is a closed system and there is no action from outside. As a result, the total momentum of box must remain constant in time, in our case **zero**. This requires that, when the wavelet leaves the source and transports the pulse ***p*** right side, the box ***must get*** an impulse (velocity ***v***) on the left side so that the moment ***vector sum remains zero***.

$$M * v - p = 0 \rightarrow M * v = p = \frac{E}{c} \qquad (16)$$

So, the box velocity is

$$v = \frac{E}{M * c} \qquad (17)$$

c) The light pulse arrives at detector location after the time interval $\Delta t = \frac{L}{c}$ (18)

During that time the center of mass of the box is shifted on the left by

$$\Delta x = v * \Delta t = \frac{E}{M * c} * \frac{L}{c} = \frac{EL}{Mc^2} \qquad (19)$$

When the detector absorbs the light pulse energy, the box receives a pulse ***p*** that compensates the linear momentum ***M*v*** owned by the box until this instant. The momentum of box becomes anew zero and it stops its movement on the left.

d) So, at the end it results that the *center of mass of box is **shifted** on the left by **Δx*** and *this happens without any action from outside the box*!!!. **This is in contradiction with physics' laws.** There only one way to solute this situation: ***We have to accept that the light pulse transfers a certain amount of mass m*** *from point S of box to the point D.*

This hypothesis avoids the perplexity if the center of mass of box would remain at rest, all time. So, we have to assume that the linear impulse of center of mass is zero

$$-M\upsilon_{Box} + m\upsilon_{light} = 0 \rightarrow -M\frac{dx_{box}}{dt} + m\frac{dx_{light}}{dt} = 0 \Rightarrow\Rightarrow -Mdx_{box} + mdx_{light} = 0 \qquad (20)$$

The equation (20) means that the sum of weighted shifts for box and pulse is zero all the time. Specially, at the interval's end, we have

$$-M*\Delta x + mL = 0 \rightarrow\rightarrow m = \frac{M*\Delta x}{L} = \frac{M}{L}*\frac{EL}{Mc^2} = \frac{E}{c^2} \qquad (21)$$

-From (21) we find the expression that ties the transported energy E and mass m as

$$E = m*c^2 \qquad (22)$$

Note that the mechanics' law operate with the inertial mass of corps and this result means that the expression (22) **relates the radiant energy contained inside an object with its inertial mass m**. As the radiation energy can be transformed into other forms of energy (thermal, chemical, electrical,.) i*t follows that the inertial mass of a body changes when it gains or loses any kind of energy*; **The inertial mass is a measure of its energy content**.

- The relativistic linear momentum expression has the same form as the classic one but the inertial mass has a relativistic meaning. So,

$$\begin{aligned} p &= m*v \\ m &= \gamma * m_0 = \frac{m_0}{\sqrt{1-(v/c)^2}} \end{aligned} \qquad (23)$$

-The rest mass $\boldsymbol{m_0}$ of a particle is measured in its rest frame. The quantity $\boldsymbol{m_0*c^2}$ is known as the ***rest energy*** of a particle. The ***relativistic kinetic energy*** of a particle is defined as the difference between the total energy $\boldsymbol{E = m*c^2}$ and its rest energy

$$K = E - m_0c^2 = (m - m_0)*c^2 = (\gamma - 1)m_0c^2 \qquad (24)$$

- Suppose we are increasing from $\boldsymbol{v = 0}$ the velocity of a particle with *rest mass* m_0 .The expressions (23) and (24) show that when $\boldsymbol{v}$ approaches to $\boldsymbol{c}$ its mass $\boldsymbol{m}$ and kinetic energy $\boldsymbol{K}$ approach the infinity (fig 5).

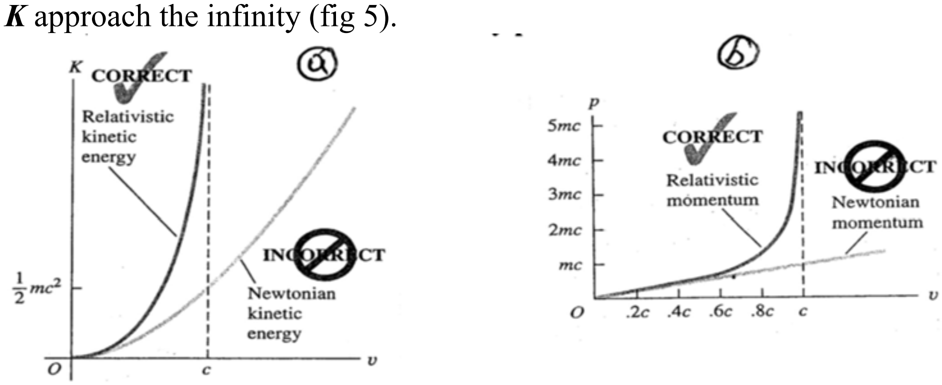


Fig 5

Here we get to a contradiction; *To increase the kinetic energy of the particle one uses the work done by a source of work. But to increase the kinetic energy of a particle to infinity one needs infinity amount of work. As there is no source with infinite reserve of energy* it comes out that it is impossible to accelerate a material particle to the speed c.
The speed of light in vacuum is unattainable by particles with a finite rest mass.

- Note that for normal velocities, one gets the classical expressions for ***m*** and ***K***.

As $\gamma = \dfrac{1}{\sqrt{1-(v/c)^2}} = [1-(v^2/c^2)]^{-1/2} \approx 1+(1/2)*(v^2/c^2)+..$ we get

$$K = (\gamma - 1)m_0c^2 \cong [1+(1/2)*(v^2/c^2)-1]m_0c^2 \cong \frac{m_0v^2}{2} \qquad (25)$$

$$m = \gamma * m_0 = [1+\frac{1}{2}*\frac{v^2}{c^2}]*m_0 \approx m_0 \qquad (26)$$

- The ***rest energy*** of a body is the sum of all "internal energies" like nuclear, atomic, chemical, thermal.. Note that from the point of view of relativity theory the mass and the energy are equivalent. In this theory the mass of an object increases when its temperature increases, the mass of a spring increases when it is compressed and so on. Similarly, the total energy of an object decreases by ***Δm*** when it releases the energy ***$\Delta E = \Delta m * c^2$.***

- The relation between the total energy and linear impulse is of special interest for different applications. Based on the following relations and by eliminating ***v*** and ***γ***

$$E = m*c^2 = \gamma * m_0 * c^2 \qquad\qquad p = \gamma * m_0 * v$$

one may find

$$\left(\frac{E}{m_0c^2}\right)^2 = \gamma^2 \qquad\qquad \left(\frac{p*c}{m_0c^2}\right)^2 = \gamma^2\frac{v^2}{c^2}$$

Then, by subtraction

$$\left(\frac{E}{m_0c^2}\right)^2 - \left(\frac{p*c}{m_0c^2}\right)^2 = \gamma^2*(1-\frac{v^2}{c^2}) = \frac{1}{(1-\frac{v^2}{c^2})}*(1-\frac{v^2}{c^2}) = 1$$

So, we get $$E^2 = p^2c^2 + (m_0c^2)^2 \qquad (26)$$

This expression relates the ***total energy*** with ***linear impulse*** and the ***rest energy***.

Note that for:

a) 'particles' with ***mo = 0*** this relation reproduces ***E = p*c*** (given by Maxwell theory).
b) 'particles' with ***mo # 0*** with very high **speed** (**close to c**) E >> $\mathbf{m_0 * c^2}$ and practically ***E ≈ p*c.*** so, when a particle has a speed close to c it behaves like an electromagnetic wave.