16.5 TRAVELLING WAVES

a) We saw the graphical way of describing pulse propagation along a string. What is the analytical way of describing this phenomenon? We start the description at a given moment that we call t = 0. We add an ***x- axis*** parallel to the string, we describe the pulse (or string) profile at ***t = 0*** by a function ***y = f(x).*** <u>This function describes the "***space behaviour***" of the pulse wave.</u>

b) To find the ***time behaviour*** we:
- consider a positive piece of **sinx**[1] profile pulse propagating along a string.
- note that each pulse ***feature***(y_1) is ***defined*** by a <u>given</u> *value* ***of phase(x_1)***.
- consider that pulse propagates ***without deformation***. This means that each pulse feature(or its <u>**phase**</u>) propagates with the <u>*same speed*</u> <u>**v**</u> along the string.
- express analytically that the ***y-value*** of pulse at given ***x- position*** at time ***t*** i.e. ***y(x, t)*** is equal to ***y-value*** of pulse at the position $x_1 = x - v^*t$ at $t = 0$ (they correspond to the same feature). So, knowing the pulse wave at t = 0 (defined by the "***space***" function ***f(x)***), we get the y-value for x position as

$$f(x, t) = f(x_1) = f(\ x - v^*t\ ) \quad (1)$$

if the pulse is travelling following positive sense of x-axis. If the pulse is moving toward the opposite sense , the *velocity is **negative*** and we get the expression

$$f(x, t) = f(x_1) = f(\ x + v^*t\ ) \quad (2)$$

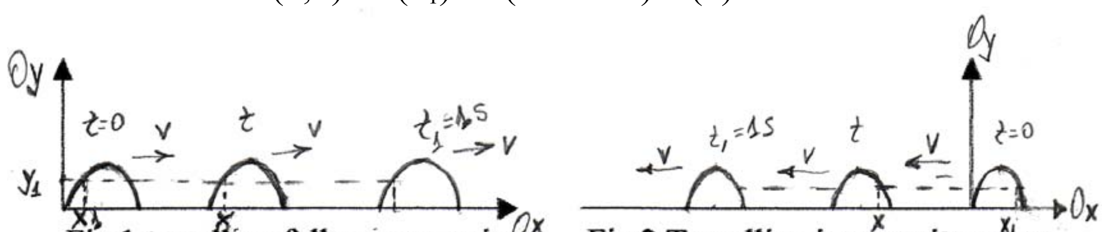

Fig.1 travelling following x-axis   Fig.2 Travelling in opposite sense

c) Each feature of pulse **at t = 0** is defined by the ***phase*** (***x***) of ***space function*** sin(x). Example the y_1 value at t = 0 is defined by the phase x_1. Note that function y = sin(x) produces the ***same feature*** for all points(***x***) & times (***t***) that fulfil the condition

$$x - v^*t = x_1 = \text{const} \quad (3) \qquad \text{or} \qquad x + v^*t = x_1 = \text{const} \quad (4)$$

The expression (***x +/- v*t***) is called the ***phase of traveling wave function*** even when the function is not trigonometric. By deriving (3,4) one gets

$$dx/dt = \pm v \qquad (5)$$

v- is called often ***wave velocity*** but strictly speaking it is the ***phase velocity***.

[1] We use a simple space function f(x) = sin(x) but it may have any kind of profile. From now on, when referring to a wave phenomena we will call ***phase*** *the quantity inside the brackets of the* ***function that describe a wave***.

d) From the practical point of view we get the function of propagating wave simply by adding the factor +/- v*t after x-argument inside the function F(x). <u>Exemple.</u> If $F(x) = \sin(2x^2)$ we get $F_+(x, t) = \sin[2(x-v^*t)^2]$ for pulse travelling versus x-positive and $F_-(x, t) = \sin[2(x + v^*t)^2]$ versus x-negative. If required to show that a function presents a propagating pulse, one has to transform it so that only factors of type (x-v*t) or (x + v*t) appear.

16.6 TRAVELLING HARMONIC WAVES

a) Consider a string under tension which one end is moved up and down in a **harmonic** way by use of an appropriate device. This device is the "source" of a traverse wave that propagates along the string. While the amplitude of this wave is limited up to certain value by the string, its period depends only on the source of oscillations. Let T be the source period (frequency f = 1/T).

b) At time T the end point has finished the first period and is ready to start the second one. Meanwhile a traverse oscillation is propagated in the string. The point where the oscillation has arrived will oscillate all time in phase (2π shift) to the end point (fig 3). The distance of this point from the string end is

$$\lambda = v^*T \qquad \text{v is the wave speed in the string} \qquad (6)$$

This distance is called the wavelength (λ) and it depends on v-value. More generally, λ is the distance between each two subsequent points of string that oscillate with 2π -phase shifts(fig 4). One says that two points oscillate "in phase" if the phase shift between them is $\Delta\phi = n^*2\pi$; n=0,+/-1,+/-2....

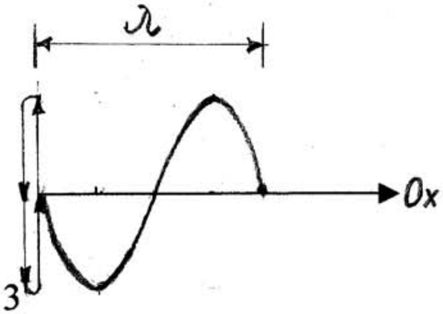

Fig 3

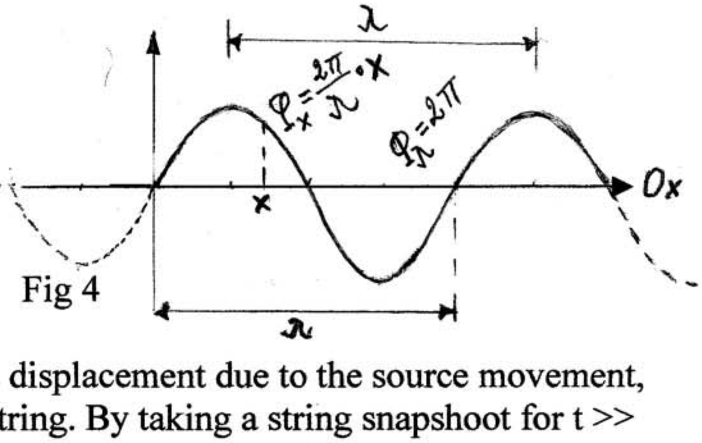

Fig 4

c) While the end point repeats its displacement due to the source movement, the oscillation travels along the string. By taking a string snapshoot for t >> T, it is easy to see a sinusoidal string profile (fig 4). This profile is described by a function of form (take amplitude = 1 for simplicity)

Such a harmonic profile may be described by a trigonometric function of form
$y(x) = \sin[\varphi(x)]$ (7) which phase " ***φ(x)*** " depends on location "***x***"

Ignoring the source[2], we put the origin of Ox axis at a point with "***disturbance***" zero and consider $\varphi(0) = 0$ so that ***y(0) = y(x=0) = sin[φ(0)] = sin(0) = 0.***
To find the phase ***φ(x)*** at a given point '***x***' we use the fact that a phase shift by 2π is produced for a ***x-shift*** equal to **λ**. The proportionality rule tells that the ***phase shift*** at the distance ***x*** is $\Delta\varphi(x) = \varphi(x) - \varphi(0) = \boldsymbol{\varphi(x) = (2\pi/\lambda)*x}$. So, the "space" wave function of the snapshot is

$$y(x) = \sin(2\pi/\lambda)*x = \sin(k*x) \qquad (7')$$

k = 2π/λ is known as the ***wave number*** (or ***wave vector***)

d) Note that the physical quantities

$$k = 2\pi/\lambda = 2\pi/(v*T) = \omega/v \qquad (8) \qquad \text{and} \qquad v = \omega / k \qquad (9)$$

concern the wave propagation as a whole in a given medium. The characteristics of motion for a ***given*** string ***point*** (**y(x)**, ***v(x), a(x)***) are derived from wave function ***for that point***. Note that *in the case a solids there are two wave velocities; one for traverse waves and one for longitudinal ones and in general they are different.*

e) To find the ***function of travelling waves*** we use the phase modification rule. So, for sinusoidal wave propagation along the positive sense of axis

$$y(x, t) = \sin[k*(x-vt)]= \sin(kx - \omega t) \qquad (10)$$

For sinusoidal wave propagation along the negative sense of axis we have

$$y(x, t) = \sin[k*(x + vt)]= \sin(kx + \omega t) \qquad (11)$$

If we consider the presence of an initial phase, these equations become

x-positive propagation $y(x, t) = \sin(kx - \omega t + \varphi)$ (12)

x- negative propagation $y(x, t) = \sin(kx + \omega t + \varphi)$ (12')

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[2] We have only a snapshot and we do not know where the source is and what direction the wave propagates.

16.7 STANDING WAVES

-Consider two simple harmonic TW waves propagating along opposite senses in the same string. Suppose they have the same amplitude and the same frequency.

Positive propagation sens : $y_r = A\sin(kx - \omega t)$ (13)

Negative propagation sens : $y_l = A\sin(kx + \omega t)$ (14)

-Superposition principle: each string particle ***disturbance*** (displacement in space).

$$y = y_l + y_r \qquad (15)$$

$$y = A\sin(kx + \omega t) + A\sin(kx - \omega t) = 2A\sin(kx)\cos(\omega t) = 2A\sin(kx)\sin(\omega t + \pi/2) \qquad (16)$$

Note that in this wave function the "***space***" factor and the "***time***" factor are ***separated***. This kind of wave is called a ***standing wave***.

-The "***space***" factor ***sin(k*x)*** defines the '***nodes***' positions [y(x, t)=0 all time] at the points where ***k*x = 0, (+/-) n*π***. As the phase difference between two ***consecutive*** nodes is ***π***, we find their Δx-distance as

$$\frac{2\pi}{\lambda}\Delta x = \pi \rightarrow\rightarrow\rightarrow \Delta x = \frac{\lambda}{2}$$

At the ***antinodes sin(k*x) = +/-1*** and their positions are found by condition ***k*x = +/- (2 n+1)*π/2***. The distance between two consecutive antinodes is

$$\frac{2\pi}{\lambda}\Delta x = \pi \rightarrow\rightarrow\rightarrow \Delta x = \frac{\lambda}{2}$$

There is one antinode between each two nodes and vice-versa (fig 5).

-The quantity $A(x) = 2A\sin(kx)$ in (16) presents the amplitude of oscillations at location ***"x"; it is a characteristic constant of this point***. In a standing wave each point of medium oscillates with "***its own amplitude***". In a traveling wave, all points of medium oscillate with same amplitude ***A***. In the two cases, the particles of medium oscillate around the equilibrium with same frequency "***ω=2πf***" (as built by source).

Note: A standing wave do not travels; the propagating waves do travel.

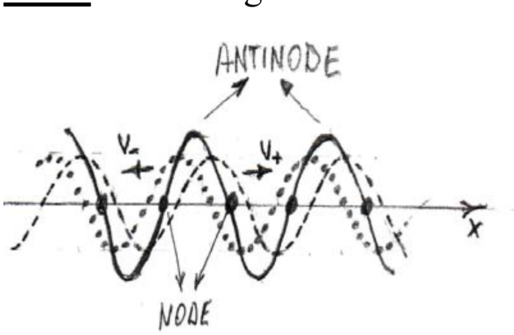


Fig 5

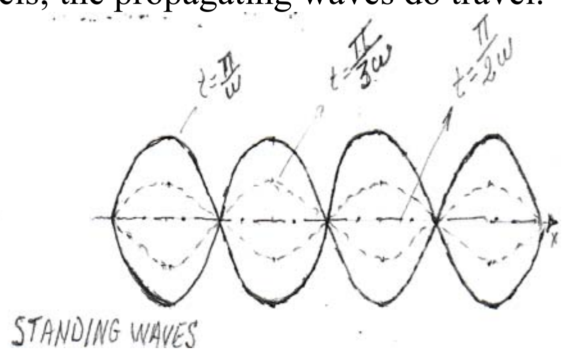


Fig 6

16.8 RESONANCE

- Standing wave equation $y = 2A\sin(kx)\cos(\omega t)$ does ***no***t put any visible ***restriction*** on ***frequency*** or ***wavelength***. This is because we did not put any ***boundary condition***. If we consider a string of ***length L*** we know that the ***end points*** of string must be all time a ***node***. So a standing wave in a finite length string must fulfill the conditions

$$\sin(kL) = 0 \rightarrow kL = n\pi \rightarrow \frac{2\pi}{\lambda}L = n\pi \Rightarrow\Rightarrow\Rightarrow \lambda = \frac{2L}{n} \qquad (17)$$

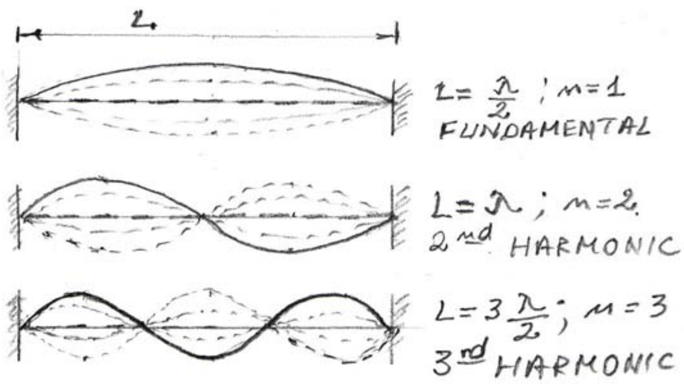
"A string with ***length L*** does not support standing waves for every λ-value. It "accepts" only those that fulfill the condition $\lambda_n = \frac{2L}{n}$ (17')

-The corresponding frequencies are known as "***resonance frequencies***"

$$\sin(kL) = 0 \rightarrow kL = n\pi \rightarrow \frac{2\pi}{v*T}L = n\pi \rightarrow \frac{\omega}{v}L = n\pi \rightarrow 2\pi f * L = n\pi v \Rightarrow f_n = n\frac{v}{2L} \qquad (18)$$

The source of waves can oscillate with different frequencies and the waves propagate in the string. When the source frequency equals one of values defined by equation (18) a standing waves appears. We say that the system source-string is vibrating "in resonance" or there is a ***resonance in*** the medium (string).

- Note that the resonance frequencies are "***string characteristics***". The first one $f_1 = \frac{v}{2L}$ is known as fundamental frequency or "first harmonic". We get the second harmonic for $f_2 = 2*\frac{v}{2L}$, and so on. The boundary conditions define the set of *string vibration modes*. The frequencies of those modes are $f_n = n*\frac{v}{2L};\_n = 1,2..$; and their wavelengths are $\lambda_n = \frac{2L}{n};\_n = 1,2,..$



Note:
These kinds of string vibration are called "***normal modes of the string***". In a more general frame, a "***normal mode of a system***" is a system motion in which all system parts move :
- With the *same frequency* and
- In a *harmonic way*.

Figure 7