15.4. SIMPLE PENDULUM, ANOTHER EXAMPLE OF SHM

-The simple pendulum is the ***first step modelling*** for the oscillations of an object tied at the end of a string. In this model, one considers the *mass "**m**" of the object concentrated in a point " **point mass**"* and the *string with no-mass* and *constant length "**L**".* Then, one selects a *positive* direction for the parameters that change with time; the *angle to the normal* "***θ***" and the distance on the arch from equilibrium "**s**" as shown in fig.1.

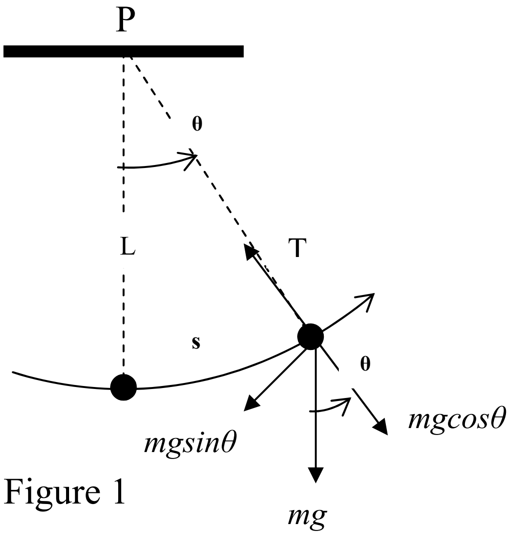


Figure 1

We apply the second law of Newton for the rotation of the point particle around an axe normal to page and passing from P-point

$$\vec{\tau}_{net} = I\vec{\alpha} \qquad (1)$$

We project eq(1) on Oz axe (normal to the page in our direction) and get

$$\tau_{net-Oz} = mL^2\alpha \qquad (2)$$

As $\alpha = \dfrac{d^2\theta}{dt^2}$ (3) and only the "***restoring** component*" of weight "***mgsinθ***" has a torque, we have $\tau_{net-Oz} = -(mg\sin\theta)*L$

and the relation (2) becomes $-mg\sin\theta * L = mL^2 * \dfrac{d^2\theta}{dt^2}$ (4)

After cancelling m, L $\quad -g\sin\theta = L*\dfrac{d^2\theta}{dt^2}$ (5)

For small angles, sin θ ~ θ and eq.(5) transforms to $-g\theta = L\dfrac{d^2\theta}{dt^2}$ or $\dfrac{d^2\theta}{dt^2} = -\dfrac{g}{L}\theta$ (6)

This equation is exactly the ***equation of a SHM*** if we pose $\boldsymbol{\theta = x}$ and $\dfrac{g}{L} = \omega^2$ (7)

This means that the ***phasor*** describing the oscillation of the "***displacement = angle θ***" rotates with *angular frequency* $\omega = \sqrt{\dfrac{g}{L}}$. Then, from the results of SHM modelling, we derive that :

- The angle θ performs harmonic oscillations described by the function

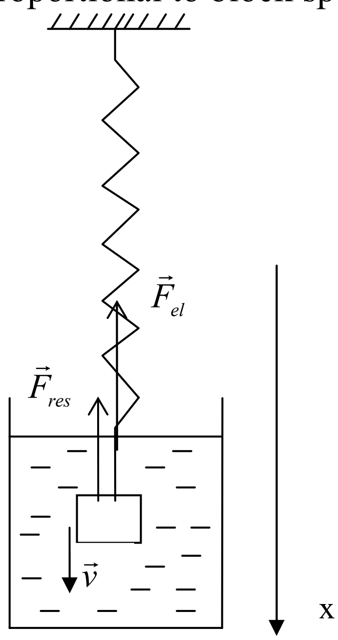
$$\theta = \theta_{max}\sin(\omega t + \varphi_0) \qquad (8)$$

- The period of real oscillations is $\quad T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{L}{g}}$ (9)

Notes ***a)*** Equation 6 tells that the "***displacement θ***" oscillates as an SHM with a period **T** given by expression (9).
b) Do not mix the ***real physical angle θ*** with the ***phase angles*** ($\varphi = \omega t + \varphi_0$, φ_0)
c) *The **linear velocity** of mass **"m"** is **v = L* dθ/dt** because **s = L*θ** (see fig.1)*

15.5 DAMPED OSCILLATIONS

-In SHO and SHM (*S- simple*) there is ***no energy loss*** with time and oscillations continue till infinity. But in a real system there is always a portion of energy that is lost and the oscillations do not continue to infinity.
-**Friction** is the main reason of energy loss in a SHM. It decreases the total energy of oscillating system in time. As $E = (1/2)k*A^2$ this appears as a *decrease of oscillations amplitude* with time.
-The motion of the spring-block system becomes visibly damped when the displacement of block is produced inside a liquid. From mechanics, we know that a *resistant force* (***drag force***, directed against to motion direction-see NYA course) is exerted on the block by the liquid. Its magnitude is proportional to block speed for ***moderate speed values***.



$$\vec{F}_{res} = -b * \vec{V} = -b\frac{dx}{dt}\vec{i} \qquad (10)$$

b[Kg/s] is the ***damping constant*** of liquid on block <u>material</u>
V[m/s] = ***dx/dt*** is the block ***velocity***, **i**-unit vector along Ox

So, $\vec{F}_{NET} = \vec{F}_{el} + \vec{F}_{res} = -kx\,\vec{i} - b\frac{dx}{dt}\vec{i} = (-kx - b\frac{dx}{dt})\,\vec{i}$ and the second law of Newton $\vec{F}_{NET} = m\,\vec{a}$ projected on Ox axis takes the form $-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$

So**, the motion equation for the free block-spring (**the weight action is canceled by the initial spring extension to equilibrium point)

$$m\frac{d^2x}{dt^2} = -kx \quad (11) \qquad \textbf{transforms to} \qquad m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} \qquad (12)$$

in presence of damping. We rewrite (12) in form

Fig 4

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \qquad (13)$$

The mathematical equation of type (13) is valid for all damped harmonic oscillations (***DHO***); mechanic, electric.., and it is very well studied. Its solution retains the general features of SHO but the oscillation amplitude changes with time; ***A = A(t).*** In the case of a damped oscillation, the "***displacement***" evolution with time takes the form;

$$x = A'\sin(\omega' t + \varphi) \quad (14)$$ where the ***damped amplitude*** is $$A' = A_0 e^{-\frac{b}{2m}t} \qquad (15)$$

and the ***damped angular frequency*** is $$\omega' = \sqrt{\omega_0^{\,2} - \left(\frac{b}{2m}\right)^2} \qquad (16)$$

<u>Remember</u> *The **<u>damped</u> angular frequency** ω' is **<u>smaller</u>** than **<u>natural</u> circular frequency*** $\omega_0 = \sqrt{\frac{k}{m}}$.

-***Underdamped* oscillations** if the <u>damped angular frequency</u> is ***a real*** number

$$\omega' > 0 \rightarrow \omega_0^{\ 2} - \left(\frac{b}{2m}\right)^2 > 0 \rightarrow \omega_0 > \left(\frac{b}{2m}\right) \rightarrow\rightarrow b < 2m\omega_0 \qquad (17) \text{ (see fig 5)}$$

(Underdamped) $A(t) = A_0 e^{-\gamma t/2m}$

The damped period is $T' = 2\pi/\omega'$.

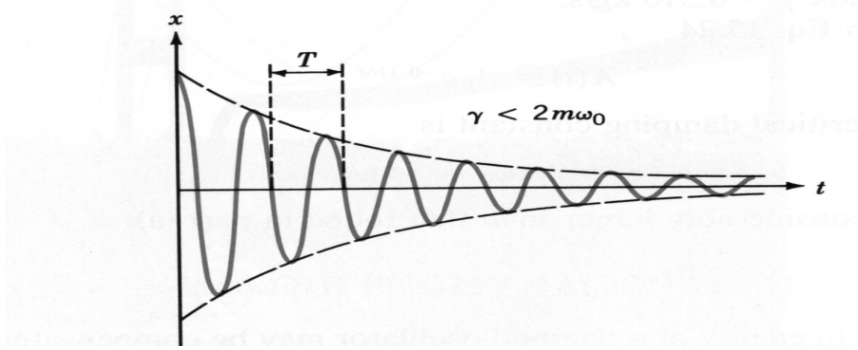


Fig 5

This is the case when oscillations are lost with the time.

Crititically-damped* motion (*no oscillations) if the angular frequency is ω’ = 0;

$$\omega' = 0 \rightarrow \omega_0^{\ 2} - \left(\frac{b}{2m}\right)^2 = 0 \rightarrow \omega_0 = \left(\frac{b}{2m}\right) \rightarrow\rightarrow b = 2m\omega_0 \qquad (18)$$

Critical damping produces a ***<u>return to equilibrium</u>*** motion with the ***<u>shortest time</u>***. (Ex: electrical device needle) If $b \leq 2m\omega_0$ the system is “*less than critical*” but <u>*not really under-damped*</u>. So, it performs 1-2 oscillations before resting at equilibrium (ex. cars’ suspension).

<u>Over-damped</u> ***motion*** *(no oscillations)* if the angular frequency ω’ is an <u>*imaginary number*</u>

$$\omega' = im \rightarrow \omega_0^{\ 2} - \left(\frac{b}{2m}\right)^2 < 0 \rightarrow \omega_0 < \left(\frac{b}{2m}\right) \rightarrow\rightarrow b > 2m\omega_0 \qquad (19)$$

In this case the system ***<u>returns to equilibrium slowly</u>***.
(ex. Doors that close slowly)

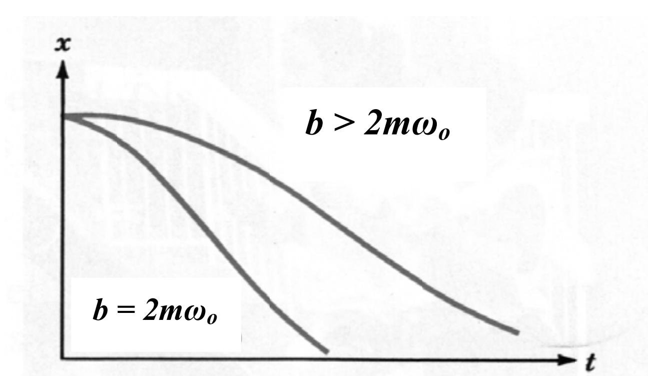


Figure 6

15.6 FORCED OSCILLATIONS

- A mechanical oscillator has its characteristic **natural angular frequency** ($\omega_0 = \sqrt{k/m}$) which corresponds to its *free oscillations* (ideal model). In reality, there is always *damping* due to the different interactions and the system looses energy. If ***b < 2mω₀*** (***under damped*** situation), it achieves a ***DHM*** with ***angular frequency*** $\omega' = \sqrt{\omega_0^2 - (b/2m)^2}$ and the oscillations *disappear* with time. If $b \geq 2m\omega_o$ *there* is an ***over** or **critically damped*** situation, there is ***no oscillation.***

- One can make oscillations continue by compensating the energy loss *in a periodic way*. To keep a ***steady-state oscillations*** one must by apply an external[1] ***periodic force.*** Note that in this case one is dealing with a ***FHO***, *forced* or a *driven* oscillation and not to ***SHM*** or ***DHM***.

Assuming that the external periodic force is $$F_{ext} = F_0 \cos\omega_{dr} * t \quad (20)$$

the equation of a *driven oscillation* undergoing damping is $$\vec{F}_{el} + \vec{F}_{res} + \vec{F}_{driv} = m\vec{a} \quad (21)$$

After projecting this equation on an axe parallel to the direction of motion (fig.4, eq.12)

one gets the expression $$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_0 \cos\omega_{dr} \quad (22)$$

which can be transformed into $$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos\omega_{dr} \quad (23)$$

The equation (23) is known as the ***equation of driven oscillations***. The solution of this equation is not the object of the course. The main qualitative results are given in the following.

A *driven oscillator* performs a harmonic motion. It has two main characteristics;

a) its *angular frequency* is equal to that of the *external driving* force ($\omega = \omega_{dr}$).

b) its ***amplitude*** depends on the *damping constant* (***b***) and ω_{dr}. *It's maximum for* $\omega_{dr} = \omega_0$.

A given oscillator has a given set of values (m, ω_0, b). While ***m*** and ω_0 are system parameters the ***b***-value depends on the surrounding mediums, too. The graphs in fig.7 (**known as resonance curves**) show the ***evolution*** of the ***amplitude of a driven oscillator*** with **the ratio of** *driving frequency* (ω_{dr}/ω_o) in three different *damping* situations. These curves present a maximum that corresponds to the situation when the **driving frequency (ω_{dr}) is close to the natural frequency (ω_0) of oscillator**. One says that *a* ***resonance*** is produced in a system when the oscillation ***amplitude*** gets the ***maximum*** value on the graph A=A(ω). The ***resonance*** of the same driven oscillator (same $k, m, \omega_0, F_o, \omega_{dr}$) is more ***pronounced*** for ***low damping (b - small)*** and may even ***disappear*** for ***high damping (b - large)***.

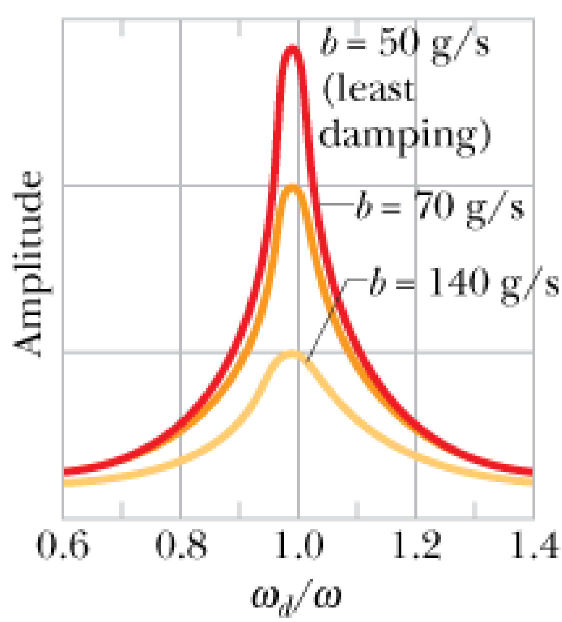


Figure 7

To simulate forced oscillations use http://www.walter-fendt.de/ph11e/resonance.htm

[1] To the oscillating system