1] OSCILLATIONS

General

-An *event* or *motion* that repeats itself at regular intervals is said to be periodic.

Periodicity in Space Ex: *The **location** of: - lines in a ruler, - slits in a diffraction grating. - atoms in the crystalline structure of a solid.*

Periodicity in time(motion) Ex: *The **position** of; - the piston in a car engine,-the block in block-spring system,- the bob in a pendulum, - voltage or the current in AC circuits*.

-In a ***periodic motion*** a the object ***oscillates above*** ("**+" *values***) and below ("**-" *values***) an **equilibrium location ("0" –value)**. By definition, an object is performing a *Simple Harmonic Motion* (***SHM***), if its "***displacement"*** *from* **equilibrium (i.e. zero** *value of* ***displacement)*** follows a **pure harmonic (sine or cosine) function**(see introduction)**.**

SHM – Function & Equation ***(*** http://www.ngsir.netfirms.com/englishhtm/SpringSHM.htm)

- The "***displacement***" of ***SHM*** evolutes in ***time*** as harmonic function. Knowing how to use the trigonometric circle for getting to the harmonic functions of ***angle***, it remains just a step for transferring the harmonic behaviour in time. We do this by using the **phasor**, a very useful physics model. Actually, the **phasor** is a **vector** with amplitude "***A-max. displacement*** " rotating at a constant[1] ***angular frequency ω*** around the origin of a frame ***Oxy*** (fig.1). The *angle-* ***phase*** and *ω-value are* taken positive for *counter clock* direction of rotation. (http://www.jelsim.org/content/applets/phasor/) If the *angle* of this vector to Ox axe is φ_0 *at* $t = 0$, then, at time "***t***" this angle will be $\varphi(t) = \varphi_0 + \omega t$. Then, using trigonometry we find that at moment " t" the *components* of this vector are

$$x(t) = A*\cos(\omega*t + \varphi) \qquad (1)$$

$$y(t) = A*\sin(\omega*t + \varphi) \qquad (1')$$

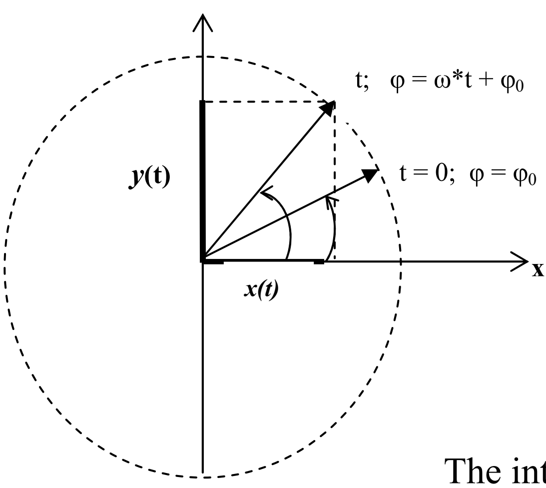

Figure 1

Each component can be used to present a SHM because each of them is an *harmonic function of time* that oscillates with angular frequency ***ω*** between the extreme displacements +A(max) and –A(min). So, one may equally chose each of them. But, once e the choice done, *one must refer to the same choice.*

In this course we will refer to y-component.

The introduced model; "***a vector with amplitude A(maximal displacement) rotating with constant angular frequency is known as the phasor model***".

Note: The ***phasor*** is the basic *model used in physics to study* ***all oscillations and waves*** *no matter the physical nature of physical parameter that oscillates***.**

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[1] Note: It is not the frequency of the real oscillation.

-After deciding to describe the "***the displacement***" SHM by using the ***y-component*** *of the phasor,* it remains to tie the ***three parameters*** of phasor model(A, ω, φ_0) to the parameters of the real displacement in time. We fix the amplitude ***A*** equal to the *maximum shift*[2] *value* from equilibrium. For ***ω*** …***one full oscillation*** of the displacement is completed in *one time period* (t = ***T***) and during this time (one period), the phasor rotates by the angle 2π. So, we get

$$\Phi = \omega * T = 2\pi \qquad (2)$$

The ***real frequency*** of oscillations of the physical quantity is ***defined as***

$$f = 1/T \qquad (3)$$

By use of (2) and (3) we get the relation between the **angular frequency**(ω) and the ***real*** frequency (***f***) of oscillations as

$$\omega = 2\pi / T = 2\pi * f \qquad (4)$$

We define φ_0 by use of the ***displacement*** value at ***t = 0***; i.e. $\varphi_0 = arsine\{y(t=0) / A\}$
In general, one prefers to use $-\pi \le \varphi_0 \le \pi$ *but taking* $\varphi_0 = -- \pi/2$ *or* $+3\pi/2$ *is the same.*

- *Notations & Dimensions*; **y**- *displacement* [positive, negative or zero];
A - ***Amplitude*** of oscillations [***positive***]; $\Phi = (\omega * t + \varphi_0)$ - ***phase*** [**r**-radian];
φ_0 - *phase constant* [**r**]; **T**- *period* [s] ; **f**- *natural freq.* [Hz =1/s]; **ω** - *angular freq.* [r/s];

- A **SHM** is an oscillation with:
  a) **Constant Amplitude** and ***constant period*** T (or frequency f).
  b) ***Harmonic*** (***sine or cosine***) time dependence for displacement.
  c) ***Constant energy*** (no energy loss ,we will explain later on)

-The ***phase constant*** may be a) $\varphi_0 = 0$; initial "displacement" = 0 (i.e. at t = 0) ;
b) $\varphi_0 \neq 0$; initial "displacement $\neq 0$ at t = 0 ;

- **Differential equation**

$$y = A * \sin(\omega * t + \varphi) \qquad (5)$$

$$y' = A\, \omega * \cos(\omega * t + \varphi) \qquad (6)$$

$$y'' = -\omega^2 [A * \sin(\omega * t + \varphi)] \qquad (7)$$

which can be presented as

$$\frac{d^2 y}{dt^2} = -\omega^2 y \_or\_ \frac{d^2 y}{dt^2} + \omega^2 y = 0 \qquad (8)$$

- As " **y** " is a *displacement in space* it comes out that **y''** = **a** is an acceleration and the relation (8) takes the form

$$a = -\omega^2 y \qquad (9)$$

Remember : the relation(8, 9) hold on ***only for a SHM*** and constitute the basic criterion for check out whether a given oscillation is a ***SHM*** or not.

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[2] Maximum positive shift from the equilibrium value

- By use of $\cos\alpha = \sin(\alpha + \pi/2)$ and $-\sin\alpha = \sin(\alpha + \pi)$, one can transform (6) to

$y' = A\omega\sin[(\omega^*t + \varphi) + \pi/2]$ (6') and (7) to $y'' = \omega^2 A^*\sin[(\omega^*t + \varphi) + \pi]$ (7')

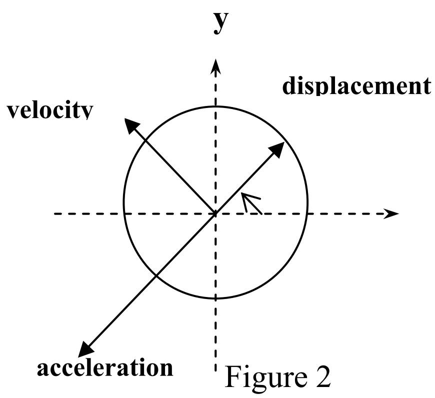

Figure 2

*Now, (6') and (7') are the y-components of two new **phasors**, **velocity phasor** with amplitude **Aω** and **acceleration phasor** with amplitude **Aω²**. The three phasors(displacement's, velocity's acceleration's) rotate with **same** angular frequency (**ω**) but the **velocity phasor** is all time **advanced** in phase **by π/2** while the **acceleration phasor** is all time **advanced** in phase by π with respect to displacement phasor (fig.3). We simply say **that** <u>**the velocity is advanced by π/2 and the acceleration advanced by π toward displacement.**</u>*

2] HORIZONTAL BLOCK-SPRING SYSTEM

- Consider a block with masse **m** and a spring *without mass*. Assume that the net force[3] applied on the block is that exerted by the spring (fig 3). The **displacement** of block from equilibrium (***x***) is equal to the spring extension. From Hook's law we know that

$$F = F_{sp}^{el} = -kx \qquad (10)$$

where k is the constant of elasticity of the spring.

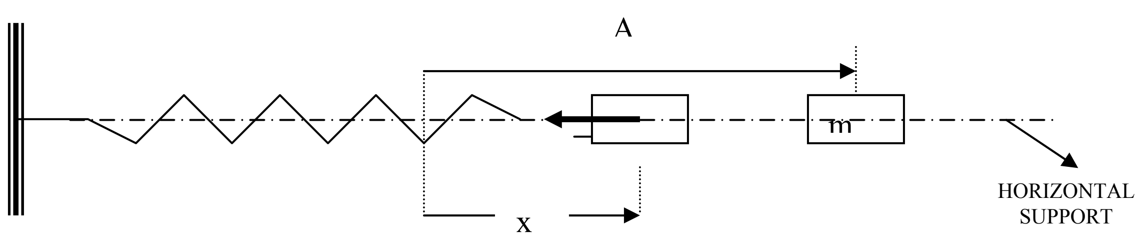

Fig 3

Applying the second law of Newton for the block movement, we get

$$F = m * a \qquad (11)$$

and by use of equation (10) we arrive at acceleration expression in form

$$a = \frac{F}{m} = \frac{F_{sp}^{elas}}{m} = -\frac{k}{m}x \qquad (12)$$

As ***k*** and ***m*** are positive quantities we can assign $\omega^2 = \frac{k}{m}$ (13)

and the relation (12) gets the form ***a = - ω² x** which is the **same as (9).***
<u>*We conclude that* the block is moving in a ***SHM***</u>. The block ***displacement*** (***x**-change*) obeys to the ***equations*** $\frac{d^2x}{dt^2} = -\frac{k}{m}x$ *or* $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$ *or* $\frac{d^2x}{dt^2} + \omega^2 x = 0$ (14)

[3] The weight of block is compensated by normal force and there is no friction between the block and the axe.

From (13) we calculate the *angular frequency* and the *period of oscillation* as

$$\omega = \sqrt{\frac{k}{m}} \qquad (15)$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad (16)$$

Note: ***T** depends on **k** and **m** but not **on A (SHM requirement).***

3] ENERGY IN SHM

-If there is no friction (*ideal model*) the net force acting on the block is equal to the ***elastic force***. From mechanics, we know that this force is a ***conservative*** force and the related mechanical potential is $U_{el} = \frac{1}{2} kx^2$. Also, the gravitational force acting on the block s conservative and it potential is $U_g = mgh$. So, we *define* the ***system spring-block – earth*** and the only external acting force is the normal force which work is zero. So, the system is an ***isolated system (*** its mechanical energy remains constant in time). The total mechanical energy of this ***isolated system is*** $E = U_{el} + U_g + K$. If we fix the zero level of energy at support, then $U_g = 0$ and the total mechanical energy of the system becomes $E = U_{el} + K$ Now, let's use the ***SHM expressions*** for ***displacement*** and ***velocity*** to calculate E.

$$x = A\sin(\omega t + \varphi) \Rightarrow\Rightarrow\Rightarrow\Rightarrow U_{el} = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\sin^2(\omega t + \varphi) \qquad (17)$$

$$x' = A\omega\cos(\omega t + \varphi) \Rightarrow\Rightarrow K = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \varphi) = \frac{1}{2}mA^2\frac{k}{m}\cos^2(\omega t + \varphi)$$

and

$$K = \frac{1}{2}A^2k\cos^2(\omega t + \varphi) \qquad (18)$$

Then

$$E = U + K = \frac{1}{2}kA^2\sin^2(\omega t + \varphi) + \frac{1}{2}A^2k\cos^2(\omega t + \varphi) = \frac{1}{2}kA^2 \qquad (19)$$

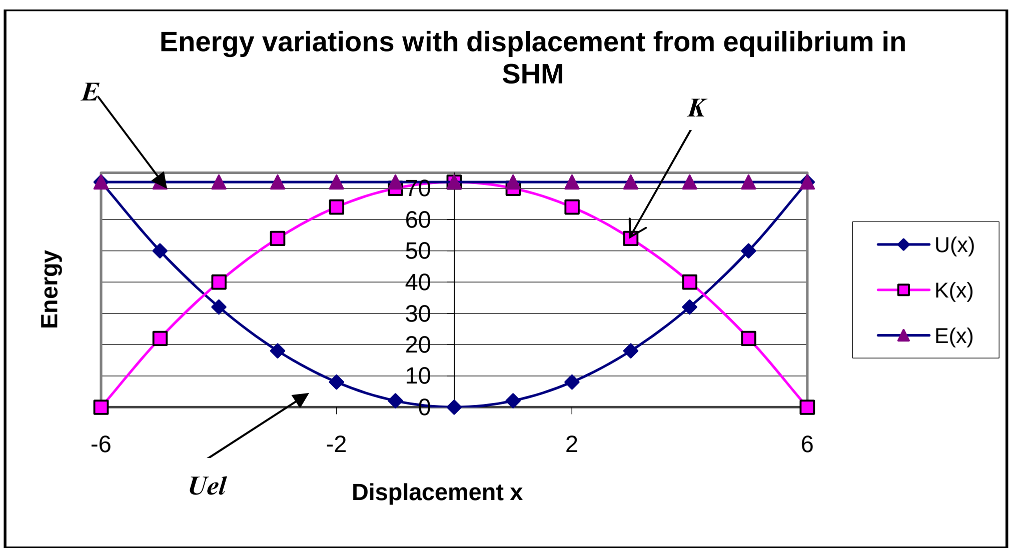

Fig 4

-As seen from equations (17- 19), ***U, K*** are <u>*positive*</u> and <u>*oscillate in time*</u> as function $\sin^2(..t)$ or $\cos^2(..t)$ but their sum, the total energy remains constant in time ($E = \frac{1}{2}kA^2$).

-The form of *potential energy* ***vs*** *displacement-**x*** is parabolic (see eq.17 and the graph). It is important to note that SHM and all SHO are characterized by a *parabolic **"potential well"***. The form of the ***potential well*** is defined by force at its origin. The parabolic form of potential well is met whenever a ***restoring*** (or ***elastic***) ***force*** is ***exerted on the system***. The ***parabolic potentials*** and ***restoring forces*** are used very often as a ***first step*** in physic's ***modelling***. Other types of forces may act on the system and the potential well is not always parabolic.

4]SIMPLE HARMONIC OSCILLATIONS

-In a *Simple Harmonic Motion* (***SHM***) the ***displacement*** is a ***shift*** in ***space***. Meanwhile, any physical parameter (current, temperature, pressure..) may *change in time* as pure harmonic function. We say that it performs a *Simple Harmonic Oscillation* (***SHO***).
So, the ***SHM*** is just a ***SHO*** where the ***displacement is a real shift in space.***

-When deriving the ***differential equation*** (8) from ***function*** (5) we did not mention any special requirement for the physical nature of ***y(t)***. Actually, the equation (8)
$\frac{d^2 y}{dt^2} = -\omega^2 y \_ or \_ \frac{d^2 y}{dt^2} + \omega^2 y = 0$ is valid for any ***SHO*** and is known as ***SHO equation***
If the "***change = displacement***" of a *physical quantity* obeys to this type of differential equation of type (8), we can affirm straight away that it is performing a ***SHO***. This is a ***basic criterion*** for verifying whether a physical quantity is performing SHO or not.

-There are ***three*** main ***types*** of HARMONIC OSCILLATIONS:

a) *Simple* Harmonic Oscillation (***SHO***); " no energy loss " = ***ideal model***
b) *Damped* Harmonic Oscillations (***DHO***); " energy loss " = ***real life model***
c) *Forced* Harmonic Oscillations (***FHO***); "external driving force compensates for lost energy" = ***real life model***

We will study DHM and FHM characteristics in next section. Note that the results are valid for any DHO and FHO because the form of mathematic expressions is the same.

Actually, almost all the results that we can get from the study of SHM are valid for SHO. Also, t*he **total energy** for any SHO has the same form as expression (19); It's only that instead of an elasticity constant "**k**" there is another "**restoring constant**".*

Important: In any SHO, the total energy $E \sim A^2$.