1 Solutions: Differentiation I

1) Differentiate the following functions with respect to x:

i) $\frac{\mathrm{d}}{\mathrm{dx}}(\cos(\pi x)) = -\pi \sin(\pi x)$

ii) $\frac{\mathrm{d}}{\mathrm{dx}}(\sin(\pi x)) = \pi \cos(\pi x)$

iii) $\frac{\mathrm{d}}{\mathrm{dx}}(\tan(\pi x)) = \pi \sec^2(\pi x)$

iv) $\frac{\mathrm{d}}{\mathrm{dx}}(2\ln(ax)) = \frac{2}{x}$

v) $\frac{\mathrm{d}}{\mathrm{dx}}\left(ae^{bx} + 1\right) = abe^{bx}$

vi) $\frac{\mathrm{d}}{\mathrm{dx}}\left(\beta(x^{\alpha n} + x)^2\right) = 2\beta(\alpha n x^{\alpha n - 1} + 1)(x^{\alpha n} + x)$

2) Differentiate the following functions in parts (**i**) to (**v**) with respect to x, and for part (**vi**) you need to differentiate with respect to t:

i) $\frac{\mathrm{d}}{\mathrm{dx}}(x\cos(\pi x) + \pi) = \cos(\pi x) - \pi x \sin(\pi x)$

ii) $\frac{\mathrm{d}}{\mathrm{dx}}\left(x^2\sin(\pi x)\right) = 2x\sin(\pi x) + \pi x^2 \cos(\pi x)$

iii) $\frac{\mathrm{d}}{\mathrm{dx}}(\ln x \tan(\pi x)) = \frac{\tan(\pi x)}{x} + \pi \ln x \sec^2(\pi x)$

iv) $\frac{\mathrm{d}}{\mathrm{dx}}\left(\beta x^2 e^{\pi x}\right) = \beta\left(2xe^{\pi x} + \pi x^2 e^{\pi x}\right)$

v) $\frac{\mathrm{d}}{\mathrm{dx}}\left(\sin(x)e^{(bx^2+cx+d)}\right) = \cos(x)e^{(bx^2+cx+d)} + \sin(x)(2bx + c)e^{(bx^2+cx+d)}$

vi) $\frac{\mathrm{d}}{\mathrm{dt}}\left((2\mathrm{t}^2 + 3\mathrm{t})\sin^{-1}(\mathrm{t})\right) = (4\mathrm{t} + 3)\sin^{-1}(\mathrm{t}) + \frac{(2\mathrm{t}^2 + 3\mathrm{t})}{\sqrt{1 - \mathrm{t}^2}}$

3) The following quotients are differentiated with respect to x:

i) $$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\cos(\pi x)}{3x^2+2x}\right) = \frac{-(3x^2+2x)\pi\sin(\pi x) - \cos(\pi x)(6x+2)}{(3x^2+2x)^2}$$

ii) $$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{x^3}{\tan(x)}\right) = \frac{3x^2\tan(x) - x^3\sec^2(x)}{\tan^2(x)}$$

iii) $$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{x\ln x}{3x^2+2x+5}\right) = \frac{(\ln x+1)(3x^2+2x+5) - x\ln x(6x+2)}{(3x^2+2x+5)^2}$$

iv) $$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\beta x^2}{1+e^{\pi x}}\right) = \frac{2\beta x(1+e^{\pi x}) - \beta\pi x^2 e^{\pi x}}{(1+e^{\pi x})^2}$$

v) $$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\sin(x)}{e^{bx}}\right) = \frac{\cos(x)e^{bx} - b\sin(x)e^{bx}}{e^{2bx}}$$

vi) $$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{2x^2+3x}{\sin^{-1}(x)}\right) = \frac{(4x+3)\sin^{-1}(x) - \frac{2x^2+3x}{\sqrt{1-x^2}}}{(\sin^{-1}(x))^2}$$

4) Implicity differentiate the function $y^2+2xy+x^3=0$ with respect to x:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dx}}\left(y^2+2xy+x^3\right) &= 0 \\
2y\frac{\mathrm{dy}}{\mathrm{dx}} + 2y + 2x\frac{\mathrm{dy}}{\mathrm{dx}} + 3x^2 &= 0 \\
2\left(x+y\right)\frac{\mathrm{dy}}{\mathrm{dx}} &= -(3x^2+2y) \\
\frac{\mathrm{dy}}{\mathrm{dx}} &= -\frac{(3x^2+2y)}{2(x+y)}
\end{aligned}$$

5) Evaluate $\frac{\mathrm{dy}}{\mathrm{dx}}$ for the parametric equation $x = 3\theta^2+2$, $y = \theta + \cos\theta$.

$$\begin{aligned}
\frac{\mathrm{dy}}{\mathrm{d}\theta} &= 1 - \sin\theta \\
\frac{\mathrm{dx}}{\mathrm{d}\theta} &= 6\theta \\
\frac{\mathrm{dy}}{\mathrm{dx}} &= \frac{\mathrm{dy}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{dx}} \\
\frac{\mathrm{dy}}{\mathrm{dx}} &= \frac{\mathrm{dy}}{\mathrm{d}\theta}\Big/\frac{\mathrm{dx}}{\mathrm{d}\theta} \\
\frac{\mathrm{dy}}{\mathrm{dx}} &= \frac{1-\sin\theta}{6\theta}
\end{aligned}$$

6) Find the equation of the tangent and normal to the curve $x = \theta^3 + 2\theta$, $y = 5\theta\sin(\pi\theta)$ at the point $\theta = \frac{1}{3}$.

The tangent is given by $y = mx - (mx_1 - y_1)$, and the normal is given by $y = \frac{-1}{m}x - (\frac{-1}{m}x_1 - y_1)$. So:

$$m = \frac{\mathrm{dy}}{\mathrm{dx}}|_{\theta=1/3}$$
$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{dx}}$$
$$\frac{\mathrm{dy}}{\mathrm{d}\theta} = 5\sin(\pi\theta) + 5\pi\theta\cos(\pi\theta)$$
$$\frac{\mathrm{dx}}{\mathrm{d}\theta} = 3\theta^2 + 2$$
$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{5\sin(\pi\theta) + 5\pi\theta\cos(\pi\theta)}{3\theta^2 + 2}$$

At $\theta = 1/3$, $m = 5(3\sqrt{3} + \pi)/14 \simeq 2.978$ $(3d.p.)$.

$$x_1 = 0.7037(4d.p.)$$
$$y_1 = 1.4434(4d.p.)$$

The equation of the tangent to the curve is

$$y = 2.978x - 0.6522$$

and the equation of the normal is

$$y = -0.3358x + 1.6797$$

7) Show that $y = A\sin(\pi x) + B\cos(\pi x)$ is a solution to the equation $\frac{\mathrm{d^2y}}{\mathrm{dx^2}} = \alpha\mathrm{y}$, and determine the value of the constant α.

$$y = A\sin(\pi x) + B\cos(\pi x)$$
$$\frac{\mathrm{dy}}{\mathrm{dx}} = \pi A\cos(\pi x) - \pi B\sin(\pi x)$$
$$\frac{\mathrm{d^2y}}{\mathrm{dx^2}} = -\pi^2 A\sin(\pi x) - \pi^2 B\cos(\pi x)$$
$$\frac{\mathrm{d^2y}}{\mathrm{dx^2}} = -\pi^2 y$$

so the function y is a solution to the equation with $\alpha = -\pi^2$.

8) Find the angle between the tangents of the curves $y = x^2 + 2$ and $y = x + 4$ at the point of intersection of the curves which has the largest value of y. The point of intersection is given by $x^2+2 = x+4$, so $(x+1)(x-2) = 0$. At $x = -1$, $y = 3$, and at $x = 2$, $y = 6$. For $y = x^2 + 2$,

$$\begin{aligned}\frac{\mathrm{dy}}{\mathrm{dx}} &= 2x = 4(\mathrm{for}\, x = 2)\\ \arctan\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right) &= \mathrm{angle}\,\theta\,\mathrm{between\,the\,tangent\,and\,x-axis}\\ \theta &= \arctan(4) = 75.96^\circ\end{aligned}$$

For $y = x + 4$

$$\begin{aligned}\frac{\mathrm{dy}}{\mathrm{dx}} &= 1\\ \theta = \arctan(1) &= 45^\circ\end{aligned}$$

The angle between the tangents of the two curves at $x = 2$ is 30.96°.

9) Evaluate the first and second derivatives of $y = \alpha\sin(\pi x)e^{-\gamma x}$.

$$\begin{aligned}y &= \alpha\sin(\pi x)e^{-\gamma x}\\ \frac{\mathrm{dy}}{\mathrm{dx}} &= \alpha\pi\cos(\pi x)e^{-\gamma x} - \alpha\gamma\sin(\pi x)e^{-\gamma x}\\ &= \alpha e^{-\gamma x}[\pi\cos(\pi x) - \gamma\sin(\pi x)]\\ \frac{\mathrm{d^2y}}{\mathrm{dx^2}} &= -\gamma\alpha e^{-\gamma x}[\pi\cos(\pi x) - \gamma\sin(\pi x)] + \alpha e^{-\gamma x}[-\pi^2\sin(\pi x) - \gamma\pi\cos(\pi x)]\\ &= \alpha e^{-\gamma x}[(\gamma^2 - \pi^2)\sin(\pi x) - 2\gamma\pi\cos(\pi x)]\end{aligned}$$

2 Solutions: Integration I

1) Evaluate the following integrals:

i) $\int \cos(\pi x)\mathrm{dx} = \frac{1}{\pi}\sin(\pi x) + C$.
ii) $\int \alpha \sin(\pi x)\mathrm{dx} = -\frac{\alpha}{\pi}\cos(\pi x) + C$.
iii) $\int \pi \sec^2(\pi x)\mathrm{dx} = \tan(\pi x) + C$.
iv) $\int \frac{2}{x}\mathrm{dx} = 2\ln|x| + C$.
v) $\int ae^{bx} + 1\mathrm{dx} = \frac{a}{b}e^{bx} + x + C$.
vi)

$$\begin{aligned}\beta\int (x^{\alpha n}+x)^2\mathrm{dx} &= \beta\int (x^{2\alpha n} + 2x^{\alpha n+1} + x^2)\mathrm{dx}, \\ &= \beta\left[\frac{x^{2\alpha n+1}}{2\alpha n+1} + \frac{2x^{\alpha n+2}}{\alpha n+2} + \frac{x^3}{3}\right] + C.\end{aligned}$$

2) Evaluate the following integrals:

i) $\int \sin(x)\cos(x)\mathrm{dx}$; Integrating by parts, taking $u = \sin x$ and $\frac{\mathrm{dv}}{\mathrm{dx}} = \cos(x)$ gives

$$\begin{aligned}\frac{\mathrm{du}}{\mathrm{dx}} &= \cos x \\ v &= \sin x \\ \int \sin(x)\cos(x)\mathrm{dx} &= uv - \int v\frac{\mathrm{du}}{\mathrm{dx}}\mathrm{dx} \\ &= \sin^2 x - \int \sin x\cos x\mathrm{dx} \\ \text{so}\int \sin(x)\cos(x)\mathrm{dx} &= \frac{1}{2}\sin^2 x + C.\end{aligned}$$

ii) $\int \alpha\frac{\sin(\pi x)}{\cos(\pi x)}\mathrm{dx}$. This is of the form $\int f'/f\mathrm{dx}$, so the solution is straightforward. $\int \alpha\frac{\sin(\pi x)}{\cos(\pi x)}\mathrm{dx} = -\frac{\alpha}{\pi}\ln|\cos(\pi x)| + C$.

iii) $\int \frac{3x^2+4x-9}{x^3+2x^2-9x}\mathrm{dx}$. This is of the form $\int f'/f\mathrm{dx}$, so

$$\int \frac{3x^2+4x-9}{x^3+2x^2-9x}\mathrm{dx} = \ln|x^3+2x^2-9x| + C$$

iv) $\int (4x+3)e^{(2x^2+3x)}\mathrm{dx}$.

$$\text{let } u = 2x^2 + 3x$$

$$\frac{\mathrm{du}}{\mathrm{dx}} = 4x + 3 \text{so substituting back we have}$$

$$\begin{aligned}\int (4x+3)e^{(2x^2+3x)}\mathrm{dx} &= \int e^u\mathrm{du} \\ &= e^u + C \\ &= e^{2x^2+3x} + C\end{aligned}$$

v) $\int \frac{abe^{bx}}{ae^{bx}+1}\mathrm{dx}$; Solve this as before, to get the solution

$$\int \frac{abe^{bx}}{ae^{bx}+1}\mathrm{dx} = \ln|ae^{bx}+1| + C$$

vi) $\int \frac{\alpha n x^{\alpha n-1}+\beta}{x^{\alpha n}+\beta x}\mathrm{dx}$. This is of the form $\int f'/f\mathrm{dx}$, so

$$\int \frac{\alpha n x^{\alpha n-1}+\beta}{x^{\alpha n}+\beta x}\mathrm{dx} = \ln|x^{\alpha n}+\beta x| + C$$

3) Evaluate the following integrals:

i) $\int \frac{3}{(x-1)(x+4)}\mathrm{dx}$.

$$\begin{aligned}
\frac{3}{(x-1)(x+4)} &= \frac{A}{x-1} + \frac{B}{x+4} \\
3 &= A(x+4) + B(x-1) \\
\text{if } x = 1 \text{ then } A &= 3/5 \\
\text{if } x = -4 \text{ then } B &= -3/5, \text{so} \\
\int \frac{3}{(x-1)(x+4)}\mathrm{dx} &= \frac{3}{5}\int \frac{1}{x-1}\mathrm{dx} - \frac{3}{5}\int \frac{1}{x+4}\mathrm{dx} \\
&= \frac{3}{5}(\ln|x-1| - \ln|x+4|) + C
\end{aligned}$$

ii) $\int \frac{1}{x^2-2x-3}\mathrm{dx}$.

$$\begin{aligned}
\frac{1}{x^2-2x-3} = \frac{1}{(x+1)(x-3)} &= \frac{A}{x+1} + \frac{B}{x-3} \\
\text{if } x = -1 \text{ then } A &= -1/4 \\
\text{if } x = 3 \text{ then } B &= 1/4, \text{so} \\
\int \frac{1}{x^2-2x-3}\mathrm{dx} &= \frac{1}{4}\int \frac{1}{x-3}\mathrm{dx} - \frac{1}{4}\int \frac{1}{x+1}\mathrm{dx} \\
&= \frac{1}{4}(\ln|x-3| - \ln|x+1|) + C
\end{aligned}$$

4) Integrate the following by parts:

i) $\int e^x \sin(x)\mathrm{dx}$;

$$\begin{aligned}
u &= e^x \\
\frac{\mathrm{dv}}{\mathrm{dx}} &= \sin x \\
\frac{\mathrm{du}}{\mathrm{dx}} &= e^x
\end{aligned}$$

$$
\begin{aligned}
v &= -\cos x \\
\int e^x \sin(x)\mathrm{dx} &= uv - \int v \frac{\mathrm{du}}{\mathrm{dx}}\mathrm{dx} \\
&= -e^x \cos x + \int e^x \cos x \mathrm{dx}
\end{aligned}
$$

Integrate by parts again

$$
\begin{aligned}
u &= e^x \\
\frac{\mathrm{dv}}{\mathrm{dx}} &= \cos x \\
\frac{\mathrm{du}}{\mathrm{dx}} &= e^x \\
v &= \sin x \\
\int e^x \sin(x)\mathrm{dx} &= -e^x \cos x + e^x \sin x - \int e^x \sin x \mathrm{dx} \\
\int e^x \sin(x)\mathrm{dx} &= \frac{e^x}{2}(\sin x - \cos x) + C
\end{aligned}
$$

ii) $\int x \cos(x)\mathrm{dx}$;

$$
\begin{aligned}
u &= x \\
\frac{\mathrm{dv}}{\mathrm{dx}} &= \cos x \\
\frac{\mathrm{du}}{\mathrm{dx}} &= 1 \\
v &= \sin x \\
\int x \cos(x)\mathrm{dx} &= x \sin x - \int \sin x \mathrm{dx} \\
&= x \sin x + \cos x + C.
\end{aligned}
$$

iii) $\int x \sec^2(x)\mathrm{dx}$.

$$
\begin{aligned}
u &= x \\
\frac{\mathrm{dv}}{\mathrm{dx}} &= \sec^2 x \\
\frac{\mathrm{du}}{\mathrm{dx}} &= 1 \\
v &= \tan x \\
\int x \sec^2(x)\mathrm{dx} &= x \tan x - \int \tan x \mathrm{dx}
\end{aligned}
$$

which we have to integrate to solve. Note that $\int \tan x \mathrm{dx} = \int \frac{\sin x}{\cos x}\mathrm{dx} = -\ln|\cos x| + C$. You can satisfy yourself that this is true by differentiating $\ln|\cos x|$.

$$
\int x \sec^2(x)\mathrm{dx} = x \tan x + \ln|\cos x| + C
$$

5) The Boltzman probability distribution for the energy spectrum of blackbody radiation at a constant temperature T is given by

$$P(E) = \frac{1}{kT}e^{-E/(kT)},$$

where k is Boltzman's constant. Calculate the average energy of the distribution which is given by

$$< E >= \frac{\int_0^\infty EP(E)\,\mathrm{dE}}{\int_0^\infty P(E)\,\mathrm{dE}}.$$

$$\begin{aligned}
\int_0^\infty P(E)\,\mathrm{dE} &= \int_0^\infty \frac{1}{kT}e^{-E/kT}\,\mathrm{dE} \\
&= -\left[e^{-E/kT}\right]_0^\infty \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty EP(E)\,\mathrm{dE} &= \int_0^\infty \frac{E}{kT}e^{-E/kT}\,\mathrm{dE} \\
&= -\left[Ee^{-E/kT}\right]_0^\infty + \int_0^\infty e^{-E/kT}\mathrm{dE} \\
&= -[0-0] - kT\left[e^{-E/kT}\right]_0^\infty \\
&= kT
\end{aligned}$$

3 Solutions: Differentiation II

1) Determine the solutions for all values of θ between 0 and 2π for the following

(**i**) $\sin^{-1}(\sqrt{3}/2), \theta = \pi/3, 2\pi/3$

(**ii**) $\cos^{-1}(1/\sqrt{2}), \theta = \pi/4, 7\pi/4$

(**iii**) $\tan^{-1}(\sqrt{3}), \theta = \pi/3, 4\pi/3$

2) Derive the expression for the derivative of $\cos^{-1}(x)$.

$$\begin{aligned} y = \cos^{-1} x, \text{so } \cos y &= x \\ \frac{\mathrm{dx}}{\mathrm{dy}} &= -\sin y \\ \frac{\mathrm{dy}}{\mathrm{dx}} &= -\frac{1}{\sin y} \\ \frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{1}{\sqrt{1-x^2}}, \text{as } \sin^2 y + \cos^2 y &= 1 \end{aligned}$$

3) Differentiate the function $y = x\sin^{-1}(x)$ with respect to x.

$$\begin{aligned} y &= x\sin^{-1}(x) \\ \frac{\mathrm{dy}}{\mathrm{dx}} &= \sin^{-1}(x) + \frac{x}{\sqrt{1-x^2}} \end{aligned}$$

4) Calculate functional form of the radius of curvature of the function $y = x - \frac{2}{x}$. Given this result, calculate the radius of curvature of the function at the point corresponding to $x = 1$.

$$\begin{aligned} R &= \frac{\left[1 + (\frac{\mathrm{dy}}{\mathrm{dx}})^2\right]^{3/2}}{\frac{\mathrm{d^2y}}{\mathrm{dx^2}}} \\ \frac{\mathrm{dy}}{\mathrm{dx}} &= 1 + \frac{2}{x^2} \\ (\frac{\mathrm{dy}}{\mathrm{dx}})^2 &= 1 + \frac{4}{x^4} + \frac{4}{x^2} \\ \frac{\mathrm{d^2y}}{\mathrm{dx^2}} &= -\frac{4}{\mathrm{x}^3} \\ R &= \frac{-x^3\left[2 + \frac{4}{x^4} + \frac{4}{x^2}\right]^{3/2}}{4} \end{aligned}$$

At $x = 1$, $R = -(2+4+4)^{3/2}/4 = -\sqrt{1000}/4 \simeq -7.91(2d.p.)$.

5) Calculate the radius of curvature of the function $y = 3x^4 - x\sin(x)$ at the point corresponding to $x = 0.5$.

$$R = \frac{\left[1 + (\frac{\mathrm{dy}}{\mathrm{dx}})^2\right]^{3/2}}{\frac{\mathrm{d^2y}}{\mathrm{dx^2}}}$$
$$\frac{\mathrm{dy}}{\mathrm{dx}} = 12x^3 - \sin x - x\cos x$$
$$\frac{\mathrm{d^2y}}{\mathrm{dx^2}} = 36\mathrm{x}^2 - 2\cos\mathrm{x} + \mathrm{x}\sin\mathrm{x}$$

At $x = 0.5$, $y' = 0.9913(4d.p.)$ and $y'' = 7.0044(4d.p.)$, so $R = 0.3986(4d.p.)$. N.B. If you computed the radius of curvature using radians, then you will have obtained: $y' = 0.5818(4d.p.)$ and $y'' = 7.4845(4d.p.)$, so $R = 0.2069(4d.p.)$.

6) The function $y = e^{-3x}\sin(x)$ describes a damped oscillator. Calculate the radius of curvature of this function at the point corresponding to $x = 1$.

$$R = \frac{\left[1 + (\frac{\mathrm{dy}}{\mathrm{dx}})^2\right]^{3/2}}{\frac{\mathrm{d^2y}}{\mathrm{dx^2}}}$$
$$y' = e^{-3x}\cos x - 3e^{-3x}\sin x$$
$$y'' = -3e^{-3x}\cos x - e^{-3x}\sin x - 3e^{-3x}\cos x + 9e^{-3x}\sin x$$
$$y'' = e^{-3x}(8\sin x - 6\cos x)$$

At $x = 1$, $y' = -0.09878(5d.p.)$ and $y'' = 0.17375(5d.p.)$, so $R = 5.83984(5d.p.)$.

7) Calculate the positions of the turning points of the function $y = x^2 + 2x + 1$, and identify the nature of the turning points. Sketch the functions y, $\frac{\mathrm{dy}}{\mathrm{dx}}$, and $\frac{\mathrm{d^2y}}{\mathrm{dx^2}}$, noting the turning points.

$$y' = 2x + 2$$
$$y'' = 2$$

The turning points are at positions that satisfy $y' = 0$, so there is a single turning point at $x = -1$. The second derivative is positive at this value of x, so this turning point is a minimum. See Figure 1 for distributions of y, y' and $y'\prime$.

8) Calculate the positions of the turning points of the function $y = e^{-x}\cos(x)$ between $x = 0$ and $x = 2\pi$. Identify the nature of the turning points and sketch the functions y, $\frac{\mathrm{dy}}{\mathrm{dx}}$, and $\frac{\mathrm{d^2y}}{\mathrm{dx^2}}$ noting the turning points.

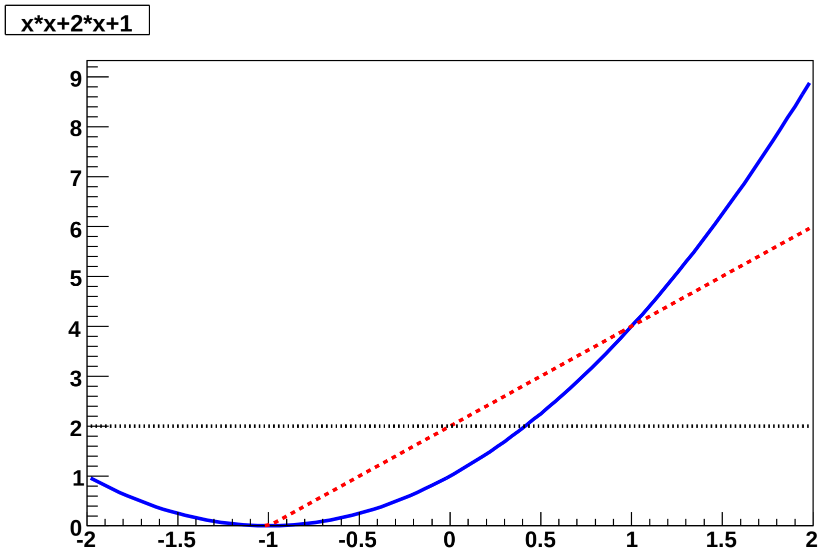

Figure 1: Distributions of y (solid), $\frac{\mathrm{dy}}{\mathrm{dx}}$ (dashed), and $\frac{\mathrm{d^2y}}{\mathrm{dx^2}}$ (dotted) for Question 7.

Just as for $\cos(x)$, there will be two turning points, and two points of inflection to determine.

$$y' = -e^{-x}(\sin(x) + \cos(x))$$
$$y'' = 2e^{-x}\sin(x)$$

The turning points are defined by $y' = 0$, which is satisfied for $\sin(x) = -\cos(x)$. Recalling that $\sin^2 x + \cos^2 x = 1$, we see that this is satisfied for $2\cos^2 x = 1$. So there are turning points at 135° (2.356 rad.) and 315° (5.497 rad.). The first of these is a minimum, the second is a maximum. The points of inflection are defined by $y'' = 0$, so when $\sin(x) = 0$. These occur at 0 and 180° (0 and π rad.). See Figure 2 for distributions of y, y' and y''.

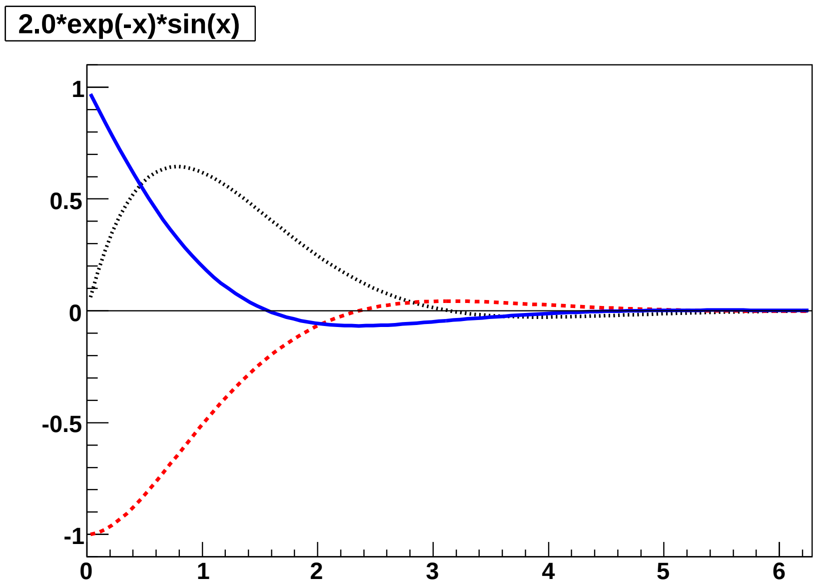

Figure 2: Distributions of y (solid), $\frac{\mathrm{dy}}{\mathrm{dx}}$ (dashed), and $\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2}$ (dotted) for Question 8.

4 Solutions: Differentiation III

1) Find all first and second partial derivatives of the function $z = x^3 + 3yx^2 - 7y$.

$$\frac{\partial z}{\partial x} = 3x^2 + 6xy$$
$$\frac{\partial^2 z}{\partial x^2} = 6x + 6y$$
$$\frac{\partial^2 z}{\partial y \partial x} = 6x$$
$$\frac{\partial z}{\partial y} = 3x^2 - 7$$
$$\frac{\partial^2 z}{\partial y^2} = 0$$
$$\frac{\partial^2 z}{\partial x \partial y} = 6x$$

2) Find all 1^{st} and 2^{nd} partial derivatives of the function $z = Ax^2 \sin(xy)$.

$$\frac{\partial z}{\partial x} = 2xA\sin(xy) + Ax^2y\cos(xy)$$
$$\frac{\partial^2 z}{\partial x^2} = (2A - Ay^2x^2)\sin(xy) + 4Axy\cos(xy)$$
$$\frac{\partial^2 z}{\partial y \partial x} = 3Ax^2\cos(xy) - Ax^3y\sin(xy)$$
$$\frac{\partial z}{\partial y} = Ax^3\cos(xy)$$
$$\frac{\partial^2 z}{\partial y^2} = -Ax^4\sin(xy)$$
$$\frac{\partial^2 z}{\partial x \partial y} = 3Ax^2\cos(xy) - Ax^3y\sin(xy)$$

3) Find all 1^{st} and 2^{nd} partial derivatives of the function $s = t^3 - 2x^2t + 7\ln(t)$.

$$\frac{\partial s}{\partial t} = 3t^2 - 2x^2 + 7/t$$
$$\frac{\partial^2 s}{\partial t^2} = 6t - 7/t^2$$
$$\frac{\partial^2 s}{\partial x \partial t} = -4x$$
$$\frac{\partial s}{\partial x} = -4xt$$

$$\frac{\partial^2 s}{\partial x^2} = -4t$$
$$\frac{\partial^2 s}{\partial t \partial x} = -4x$$

4) Calculate the total differential of $z = y \ln x + 3x \sin y$.

$$\delta z = \frac{\partial z}{\partial y}\delta y + \frac{\partial z}{\partial x}\delta x$$
$$\frac{\partial z}{\partial y} = \ln x + 3x \cos y$$
$$\frac{\partial z}{\partial x} = y/x + 3 \sin y$$
$$\delta z = (\ln x + 3x \cos y)\, \delta y + (y/x + 3 \sin y)\, \delta x$$

5) Calculate the first and second partial derivatives with respect to θ and ϕ of the function defined by $z = 2x + y$, where $x = g(\theta, \phi)$, and $y = h(\theta, \phi)$. Then evaluate these derivatives given that $x = \theta^3 + 3\phi$ and $y = \theta \sin(\phi)$.

$$\frac{\partial z}{\partial \theta} = 6\theta^2 + \sin \phi$$
$$\frac{\partial^2 z}{\partial \theta^2} = 12\theta$$
$$\frac{\partial^2 z}{\partial \phi \partial \theta} = \cos \phi$$
$$\frac{\partial z}{\partial \phi} = 6 + \theta \cos \phi$$
$$\frac{\partial^2 z}{\partial \phi^2} = -\theta \sin \phi$$
$$\frac{\partial^2 z}{\partial \theta \partial \phi} = \cos \phi$$

6) The Boltzman probability distribution for the energy spectrum of blackbody radiation at a given temperature T is

$$P(E, T) = \frac{1}{kT} e^{-E/(kT)},$$

where k is Boltzman's constant. Calculate the total differential of $P(E, T)$ and hence $\delta P/P$.

$$\delta P = \frac{\partial P}{\partial T}\delta T + \frac{\partial P}{\partial E}\delta E$$

$$\begin{aligned}
\frac{\partial \mathrm{P}}{\partial \mathrm{T}} &= \frac{\partial \mathrm{P}}{\partial \mathrm{u}}\frac{\partial \mathrm{u}}{\partial \mathrm{T}} \text{ where } u = \frac{1}{kT} \\
&= \left([1 - E/(kT)]e^{-E/kT}\right)\left(-\frac{1}{kT^2}\right) \\
\frac{\partial \mathrm{P}}{\partial \mathrm{E}} &= -\frac{1}{(kT)^2}e^{-E/kT} \\
\delta P &= \left(\left[\frac{E}{k^2T^3} - \frac{1}{kT^2}\right]\delta T - \frac{\delta E}{k^2T^2}\right)e^{-E/kT} \\
\frac{\delta P}{P} &= \left[\frac{E}{kT^2} - \frac{1}{T}\right]\delta T - \frac{\delta E}{kT}
\end{aligned}$$

7) The tilt angle ψ of the transverse profile of an positron beam in a storage ring is related to the transverse beam sizes σ_x and σ_y through the following equation:

$$\tan(2\psi) = f(\sigma_x, \sigma_y) = \frac{2\sigma_{xy}}{\sigma_x^2 + \sigma_y^2},$$

where the the $x-y$ coupling parameter σ_{xy} is assumed to be constant for a given point on the ring. Calculate the total differential δf. In the PEP-II storage ring, the beam sizes of the positron ring are $\sigma_x = 100\mu m$ and $\sigma_y = 5\mu m$. If the measured value of σ_x changes by 1%, σ_y changes by 0.5%, and $\sigma_{xy} = 0.1$, calculate δf.

$$\delta f = \frac{\partial \mathrm{f}}{\partial \sigma_{\mathrm{x}}}\delta\sigma_x + \frac{\partial \mathrm{f}}{\partial \sigma_{\mathrm{y}}}\delta\sigma_y$$

$$\frac{\partial \mathrm{f}}{\partial \sigma_{\mathrm{x}}} = -\frac{4\sigma_{xy}\sigma_x}{(\sigma_x^2 + \sigma_y^2)^2}$$

$$\frac{\partial \mathrm{f}}{\partial \sigma_{\mathrm{y}}} = -\frac{4\sigma_{xy}\sigma_y}{(\sigma_x^2 + \sigma_y^2)^2}$$

$$\delta f = -\frac{4\sigma_{xy}}{(\sigma_x^2 + \sigma_y^2)^2}\left(\sigma_x\delta\sigma_x + \sigma_y\delta\sigma_y\right)$$

Given the values of σ_x, σ_y, σ_{xy}, $\delta\sigma_x$, and $\delta\sigma_y$ above, we can calculate

$$\begin{aligned}
\delta f &= -\frac{0.4}{(100^2 + 5^2)^2}(100 * 1 + 5 * 0.025) \\
&= -3.985\times 10^{-7}
\end{aligned}$$

5 Solutions: Series

Note the following are useful (alternatively you can construct the expression for a series from first principle):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

1) The proof of the sum of n terms in an AP is;

$$\begin{aligned}
S_n &= a + (a+d) + (a+2d) + (a+3d) + \ldots + (a+[n-1]d) \\
2S_n &= a + (a+d) + (a+2d) + (a+3d) + \ldots + (a+[n-1]d) \\
&\quad +(a+[n-1]d) + (a+[n-2]d) + \ldots + a \\
&= (2a+[n-1]d) + (2a+[n-1]d) + \ldots + (2a+[n-1]d) \\
&= n(2a+[n-1]d) \\
S_n &= \frac{n}{2}(2a+[n-1]d)
\end{aligned}$$

2) The proof of the sum of n terms in a GP is;

$$\begin{aligned}
S_n &= a + ar + ar^2 + ar^3 + \ldots + ar^{(n-1)} \\
rS_n &= ar + ar^2 + ar^3 + \ldots + ar^n \\
S_n - rS_n &= a - ar^n \\
S_n &= \frac{a(1-r^n)}{1-r}
\end{aligned}$$

3) Write down all terms of the series $\sum_{i=1}^{5} x(x+1)^i$.

$$\sum_{i=1}^{5} x(x+1)^i = x(x+1) + x(x+1)^2 + x(x+1)^3 + x(x+1)^4 + x(x+1)^5.$$

4) A Maclaurin series is given by

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f''''(0)\frac{x^4}{4!} + \ldots$$

i) The maclaurin series for $x\sin(x)$ is built from the following;

$$\begin{aligned} f(x) &= x\sin(x), & \text{so,} \quad & f(0) = 0 \\ f^I(x) &= x\cos(x) + \sin(x), & \text{so,} \quad & f^I(0) = 0 \\ f^{II}(x) &= 2\cos(x) - x\sin(x), & \text{so,} \quad & f^{II}(0) = 2 \\ f^{III}(x) &= -3\sin(x) - x\cos(x), & \text{so,} \quad & f^{III}(0) = 0 \\ f^{IV}(x) &= -4\cos(x) + x\sin(x), & \text{so,} \quad & f^{IV}(0) = -4 \\ f^{V}(x) &= 5\sin(x) + x\cos(x), & \text{so,} \quad & f^{V}(0) = 0 \\ f^{VI}(x) &= 6\cos(x) - x\sin(x), & \text{so,} \quad & f^{VI}(0) = 6 \end{aligned} \tag{1}$$

So we can write the Maclaurin expansion for $x\sin(x)$ as $f(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} + \ldots$.

ii) $f(x) = e^x \sin(x)$, again write down the first three non-zero terms of the Maclaurin Series expansion for this function.

$$\begin{aligned} f(x) &= e^x\sin(x), & \text{so,} \quad & f(0) = 0 \\ f^I(x) &= e^x(\cos(x) + \sin(x)), & \text{so,} \quad & f^I(0) = 1 \\ f^{II}(x) &= 2e^x\cos(x), & \text{so,} \quad & f^{II}(0) = 2 \\ f^{III}(x) &= 2e^x(\cos(x) - \sin(x)), & \text{so,} \quad & f^{III}(0) = 2 \end{aligned} \tag{2}$$

and we can write the approximation for $f(x)$ as

$$f(x) \simeq x + x^2 + \frac{x^3}{3}$$

If you start by multiplying together the known Maclaurin series expansions for the functions e^x and $\sin(x)$, you will end up with the same result.

iii) $f(x) = (x^2+1)e^x$, again write down the first three non-zero terms of the Maclaurin series expansion for $f(x)$. The Maclaurin series expansion for e^x is known:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

and from this we can work out

$$x^2e^x = x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} + \ldots$$

So,

$$(1+x^2)e^x = 1 + x + \frac{3x^2}{2!} + \frac{7x^3}{3!} + \ldots$$

If you try to derive the series using the usual method, you will obtain the same answer shown above.

5) The general form of a Taylor series expansion is

$$f(x) = f(a)+f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2!}+f'''(a)\frac{(x-a)^3}{3!}+f''''(a)\frac{(x-a)^4}{4!}+\ldots$$

and the first four non-zero terms of the Taylor series expansion about $x = \pi/3$ for $\cos(x)$ is given by

$$\begin{aligned} f(x) &= \cos(x), \quad \text{so,} \quad f\left(\frac{\pi}{3}\right) = \frac{1}{2} \\ f^{I}(x) &= -\sin(x), \quad \text{so,} \quad f^{I}\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \\ f^{II}(x) &= -\cos(x), \quad \text{so,} \quad f^{II}\left(\frac{\pi}{3}\right) = -\frac{1}{2} \\ f^{III}(x) &= \sin(x), \quad \text{so,} \quad f^{III}\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \end{aligned}$$

So we can write the Taylor series expansion as

$$\cos(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2}\frac{(x-\frac{\pi}{3})^2}{2!} + \frac{\sqrt{3}}{2}\frac{(x-\frac{\pi}{3})^3}{3!} + \ldots$$

i)Using this expansion, we can estimate the value of $\cos(\frac{\pi}{4})$;

$$\begin{aligned} \cos\left(\frac{\pi}{4}\right) &\simeq \frac{1}{2} - \frac{\sqrt{3}}{2}\left(-\frac{\pi}{12}\right) - \frac{1}{2}\frac{(-\frac{\pi}{12})^2}{2!} + \frac{\sqrt{3}}{2}\frac{(-\frac{\pi}{12})^3}{3!} \\ &\simeq \frac{1}{2} + 0.22672 - 0.01713 - 0.00259 \\ &\simeq 0.70700 \end{aligned}$$

whereas the full series expansion (or a calculator) gives a result of 0.707107.

ii) Using this expansion, we can estimate the value of $\cos(1.0)$;

$$\begin{aligned} \cos(1.0) &\simeq \frac{1}{2} - \frac{\sqrt{3}}{2}(-0.0471976) - \frac{1}{2}\frac{(-0.0471976)^2}{2!} + \frac{\sqrt{3}}{2}\frac{(-0.0471976)^3}{3!} \\ &\simeq 0.5 + 0.040874 - 0.000557 - 0.000015 \\ &\simeq 0.540302 \end{aligned}$$

and using the full series expansion gives the same result at this level of precision.

6) Write down the first four non-zero terms of the Taylor series expansions of

i) e^x about $x = 1$; The Taylor series expanded about a point $x = a$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

so for $f(x) = e^x$, with $a = 1$, we have

$$\begin{aligned} f(x) &= e^x \\ f'(x) &= e^x \\ f^n(x) &= e^x, \end{aligned}$$

so

$$f(1) = e$$
$$f'(1) = e$$
$$f^n(1) = e$$

, and thus

$$f(x) \simeq e\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots\right)$$

You can check if the result is sensible by substituting in $x = 1$ to the above expression, and you get back e as expected, and similarly for values of x near 1.

ii) $e^x \cos(x)$ about $x = \frac{\pi}{2}$;

$$\begin{aligned}
f(x) &= e^x \cos x \\
f'(x) &= e^x(\cos x - \sin x) \\
f''(x) &= -2e^x \sin x \\
f'''(x) &= -2e^x(\cos x + \sin x) \\
f''''(x) &= -4e^x \cos x \\
f'''''(x) &= -4e^x(\cos x - \sin x),
\end{aligned}$$

so

$$\begin{aligned}
f(\pi/2) &= 0 \\
f'(\pi/2) &= -e^{\pi/2} \\
f''(\pi/2) &= -2e^{\pi/2} \\
f'''(\pi/2) &= -2e^{\pi/2} \\
f''''(\pi/2) &= 0 \\
f'''''(\pi/2) &= 4e^{\pi/2}.
\end{aligned}$$

Hence we can write the first four non-zero terms of the Taylor series expansion as,

$$f(x) \simeq e^{\pi/2}\left(-(x-\pi/2) - (x-\pi/2)^2 - \frac{(x-\pi/2)^3}{3} + \frac{1}{30}(x-\pi/2)^5\right).$$

7) Write down the general form of the binomial series expansion:

$$f(x) = (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

Using the first three terms of a binomial series expansion, estimate the following quantities and compare with the results obtained from your calculator:

i) $\sqrt{0.95} \simeq 1 - 0.05/2 - 0.0025/8 = 0.97469(5d.p.)$ which compares to 0.97468 ($5d.p.$) from a calculator.

ii) $\sqrt{1.15} \simeq 1 + 0.15/2 - 0.0225/8 = 1.07219(5d.p.)$ which compares to 1.07238 ($5d.p.$) from a calculator.

6 Solutions: Complex Numbers

1)

i) $i^2 = -1$,
ii) $i^5 = i$,
iii) $3i^{11} = -3i$,
iv) $i^{13} = i$.

2)

i) $(2+3i)+(5+7i) = 7+10i$,
ii) $(2+3i)-(5+7i) = -3-4i$,
iii) $(2+3i)(2-3i) = 4+6i-6i-9i^2 = 13$,
iv) $(3-5i)^3 = (3-5i)(9-30i+25i^2) = (3-5i)(-16-30i) = -198-10i$,
v) $\frac{3-5i}{-1-2i} = \frac{(3-5i)(-1+2i)}{(-1-2i)(-1+2i)} = \frac{1}{5}(-3+6i+5i-10i^2) = (7+11i)/5$.

3)

i) $7+10i = \sqrt{149}e^{0.9599i}$,
ii) $-3-4i = \sqrt{25}e^{4.07i}$,
iii) $13 = 13e^{0i}$,
iv) $-198-10i = 198.25e^{3.192i}$,
v) $(-7-11i)/5 = \frac{\sqrt{170}}{5}e^{1.004i}$.

4) $(-2-5i)(3-2i) = (-6+4i-15i+10i^2) = -16-11i$.

5) $\left(\frac{1}{2}-\frac{1}{\sqrt{3}}i\right)\left(-\frac{1}{2}+\frac{1}{\sqrt{2}}i\right) = -\frac{1}{4}+\frac{i}{2\sqrt{2}}+\frac{i}{2\sqrt{3}}-\frac{i^2}{\sqrt{6}} = \left(-\frac{1}{4}+\frac{1}{\sqrt{6}}\right)+\left(\frac{1}{2\sqrt{3}}+\frac{1}{2\sqrt{2}}\right)i$.

6) $(6-2i)(1-i)(2-2i) = (6-2i)(2-4i+2i^2) = -4i(6-2i) = -8-24i$.

7) $\frac{1}{i}\frac{6-2i}{1-i} = \frac{(6-2i)(1-i)}{(1+i)(1-i)} = \frac{6-6i-2i+2i^2}{2} = 2-4i$.

8) $e^{2+i\pi/4} = e^2e^{i\pi/4} = e^2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = e^2\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = 5.225(1+1i)$ ($3d.p.$).

9) $3-2i = re^{i\theta} = r(\cos\theta+i\sin\theta)$. $r = \sqrt{(3^2+2^2)} = \sqrt{13}$, and $\theta = \arg(\mathrm{z}) = \arctan(-2/3)$, so $\theta = 5.695$ ($3d.p.$). So we can write the complex number as $\sqrt{13}(\cos 5.695 + i\sin 5.695)$.

10) If $z = 4(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$, then the square roots of z are $\sqrt{z} = 2e^{i\pi/6}$, $2e^{i7\pi/6}$. The principle root is the one closest to the real axis, which is $\sqrt{z} = 2e^{i\pi/6}$.

11) $z = 3(\cos\pi/6+i\sin\pi/6)$, so $z^4 = 3^4e^{i4\pi/6} = 81(\cos\frac{4\pi}{6}+i\sin\frac{4\pi}{6})$ in polar form.

12) From question 10, $z = 3 - 2i = \sqrt{13}e^{326.3^{\circ}i} = \sqrt{13}e^{5.6950i}$, so $z^{\frac{1}{5}} = \sqrt[10]{13}e^{i(5.6950+2\pi m)/5}$, where $m = 0, 1, 2, 3, 4$ are the roots of z for $0 < \arg(\mathrm{z}) < 2\pi$. So the solutions are: $z^{\frac{1}{5}} = \sqrt[10]{13}e^{1.139i}$, $\sqrt[10]{13}e^{2.396i}$, $\sqrt[10]{13}e^{3.653i}$, $\sqrt[10]{13}e^{4.9096i}$, $\sqrt[10]{13}e^{6.1662i}$, where the last of these is the principle root.

13) As $x(x+y)+x-yi = -1-3i$, we can match the real and imaginary parts, giving $x^2+xy+x = -1$ and $y = 3$. So $x^2+4x = -1$ and therefore $x = -2 \pm \sqrt{3}$.

14) Starting from the amplitudes

$$A_1 = ae^{i\phi_1},$$
$$A_2 = be^{i\phi_2},$$

we can calculate the total probability amplitude as $A = A_1 + A_2 = ae^{i\phi_1} + be^{i\phi_2}$. So the probability is given by

$$\begin{aligned}
P(A) &= AA^* \\
&= (ae^{i\phi_1} + be^{i\phi_2}) \times (ae^{i\phi_1} + be^{i\phi_2})^* \\
&= (ae^{i\phi_1} + be^{i\phi_2}) \times (ae^{-i\phi_1} + be^{-i\phi_2}) \\
&= a^2e^{i(\phi_1-\phi_1)} + abe^{i(\phi_1-\phi_2)} + abe^{i(\phi_2-\phi_1)} + b^2e^{i(\phi_2-\phi_2)} \\
&= a^2 + abe^{i(\phi_1-\phi_2)} + abe^{i(\phi_2-\phi_1)} + b^2
\end{aligned}$$

If $\phi_1 = \phi_2$, then $P(A) = a^2 + 2ab + b^2$.

7 Solutions: Integration II

1) Determine the reduction formula for $I_n = \int x^n \sin(\gamma x)\mathrm{dx}$. Hence evaluate $I_3 = \int x^3 \sin x \mathrm{dx}$. In order to determine the reduction formula, we need to integrate by parts.

$$u = x^n$$
$$\frac{\mathrm{dv}}{\mathrm{dx}} = \sin(\gamma x)$$
$$\frac{\mathrm{du}}{\mathrm{dx}} n x^{n-1}$$
$$v = -\frac{1}{\gamma}\cos(\gamma x)$$
$$I_n = -\frac{x^n}{\gamma}\cos(\gamma x) + \frac{n}{\gamma}\int x^{n-1}\cos(\gamma x)\mathrm{dx}$$

and we can integrate $\int x^{n-1}\cos(\gamma x)\mathrm{dx}$ by parts as follows

$$u = x^{n-1}$$
$$\frac{\mathrm{dv}}{\mathrm{dx}} = \cos(\gamma x)$$
$$\frac{\mathrm{du}}{\mathrm{dx}} = (n-1)x^{n-2}$$
$$v = \frac{1}{\gamma}\sin(\gamma x)$$
$$\int x^{n-1}\cos(\gamma x)\mathrm{dx} = \frac{x^{n-1}\sin\gamma x}{\gamma} - \frac{n-1}{\gamma}\int x^{n-2}\sin(\gamma x)\mathrm{dx}$$

so

$$I_n = -\frac{x^n}{\gamma}\cos(\gamma x) + \frac{n x^{n-1}\sin(\gamma x)}{\gamma^2} - \frac{n(n-1)}{\gamma^2} I_{n-2}$$

So this can now be applied to determine I_3:

$$I_3 = -\frac{x^3}{\gamma}\cos(\gamma x) + \frac{3x^2\sin(\gamma x)}{\gamma^2} - \frac{6}{\gamma^2} I_1$$
$$I_1 = -\frac{x^1}{\gamma}\cos(\gamma x) + \frac{x^0\sin(\gamma x)}{\gamma^2} - 0$$
$$I_3 = \left(-\frac{x^3}{\gamma} + \frac{6x}{\gamma^3}\right)\cos(\gamma x) + \left(\frac{3x^2}{\gamma^2} - \frac{6}{\gamma^4}\right)\sin(\gamma x) + C$$

So as $\gamma = 1$,

$$I_3 = \left(-x^3 + 6x\right)\cos(x) + \left(3x^2 - 6\right)\sin(x) + C.$$

2) Determine the reduction formula for $I_n = \int x^n e^{i\gamma x} \mathrm{dx}$. Hence evaluate $I_2 = \int x^2 e^{i\gamma x} \mathrm{dx}$.

$$u = x^n$$
$$\frac{\mathrm{dv}}{\mathrm{dx}} = e^{i\gamma x}$$
$$\frac{\mathrm{du}}{\mathrm{dx}} n x^{n-1}$$
$$v = \frac{e^{i\gamma x}}{i\gamma}$$

$$\begin{aligned} I_n &= \frac{x^n e^{i\gamma x}}{i\gamma} - \frac{n}{i\gamma}\int x^{n-1} e^{i\gamma x} \mathrm{dx} \\ &= \frac{x^n e^{i\gamma x}}{i\gamma} - \frac{n}{i\gamma} I_{n-1} \end{aligned}$$

So

$$\begin{aligned} I_2 &= \frac{x^2 e^{i\gamma x}}{i\gamma} - \frac{2}{i\gamma} I_1 \\ I_1 &= \frac{x e^{i\gamma x}}{i\gamma} - \frac{1}{i\gamma}\int e^{i\gamma x} \mathrm{dx} \end{aligned}$$

so

$$I_2 = \frac{x^2 e^{i\gamma x}}{i\gamma} - \frac{2x e^{i\gamma x}}{(i\gamma)^2} + \frac{2 e^{i\gamma x}}{(i\gamma)^3} + C$$

3) Evaluate the integral $\int \sin^3 x \mathrm{dx}$.

$$\begin{aligned} \int \sin^3 x \mathrm{dx} &= \int \sin x (1 - \cos^2 x) \mathrm{dx} \\ &= \int \sin x dx - \int \sin x \cos^2 x \mathrm{dx} \end{aligned}$$

Now if we integrate $\sin x \cos^2 x$ by parts,

$$\begin{aligned} u &= \cos^2 x \\ \frac{\mathrm{du}}{\mathrm{dx}} &= -2 \sin x \cos x \\ \frac{\mathrm{dv}}{\mathrm{dx}} &= \sin x \\ v &= -\cos x \\ \int \sin x \cos^2 x \mathrm{dx} &= -\cos^3 x - 2\int \sin x \cos^2 x \mathrm{dx} \\ &= -\frac{1}{3}\cos^3 x \end{aligned}$$

So

$$\int \sin^3 x\mathrm{dx} = -\cos x + \frac{1}{3}\cos^3 x + C$$

4) Evaluate the integral $\int \cos^4 x\mathrm{dx}$.

$$\begin{aligned}\int \cos^4 x\mathrm{dx} &= \frac{1}{4}\int(1+\cos 2x)(1+\cos 2x)\mathrm{dx}\\ &= \frac{1}{4}\int 1+2\cos 2x+\cos^2 2x\mathrm{dx}\\ &= \frac{1}{4}\int \frac{3}{2}+2\cos 2x+\frac{1}{2}\cos 4x\mathrm{dx}\\ &= \frac{3x}{8}+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}+C\end{aligned}$$

5) Evaluate the integral $\int_{\theta=0}^{\theta=\pi/2} y\mathrm{dx}$, where $x = \sin\theta$ and $y = 0.5\cos\theta$

$$\begin{aligned}\frac{\mathrm{dx}}{\mathrm{d}\theta} &= \cos\theta,\text{ so dx} = \cos\theta\mathrm{d}\theta\\ \int_{\theta=0}^{\theta=\pi/2} y\mathrm{dx} &= \frac{1}{2}\int_{\theta=0}^{\theta=\pi/2}\cos^2\theta\mathrm{d}\theta\\ &= \frac{1}{4}\int_{\theta=0}^{\theta=\pi/2} 1+\cos 2\theta\mathrm{d}\theta\\ &= \frac{1}{4}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2}\\ &= \frac{\pi}{8}\end{aligned}$$

6) Evaluate the integral $\int_{\theta=0}^{\theta=\pi} y\mathrm{dx}$, where $x = a^\theta$ and $y = 1/(a^\theta \ln a)$, where a is a constant.

$$\frac{\mathrm{dx}}{\mathrm{d}\theta} = a^\theta \ln a$$

$$\begin{aligned}\int_{\theta=0}^{\theta=\pi} y\mathrm{dx} &= \int_{\theta=0}^{\theta=\pi}\frac{1}{a^\theta\ln a}\mathrm{dx}\\ &= \int_{\theta=0}^{\theta=\pi}\frac{a^\theta\ln a}{a^\theta\ln a}\mathrm{d}\theta\\ &= \pi\end{aligned}$$

7) Calculate the area bounded by the curve $y = \frac{1}{x} + 3\sin x$ and the x axis between $x = 1$ and $x = 2$.

$$\begin{aligned} I &= \int_{x=1}^{x=2} \int_{y=0}^{y=\frac{1}{x}+3\sin x} \mathrm{dydx} \\ &= \int_{x=1}^{x=2} \frac{1}{x} + 3\sin x \mathrm{dx} \\ &= [\ln|x| - 3\cos x]_{x=1}^{x=2} \\ &= (\ln 2 - 3\cos 2) - (\ln 1 - 3\cos 1) \\ &= 0.69452(5d.p.) \end{aligned}$$

8) Calculate the area bounded by the x axis and the curve $y = e^{-x/\pi}\sin(x)$ between $x = 0$ and $x = 2\pi$.

$$\begin{aligned} I &= \int_{x=0}^{x=2\pi} \int_{y=0}^{y=e^{-x/\pi}\sin(x)} \mathrm{dydx} \\ &= \int_{x=0}^{x=2\pi} e^{-x/\pi}\sin(x)\mathrm{dx} \end{aligned}$$

which we can integrate by parts

$$\begin{aligned} u &= e^{-x/\pi} \\ \frac{\mathrm{du}}{\mathrm{dx}} &= -\frac{1}{\pi}e^{-x/\pi} \\ \frac{\mathrm{dv}}{\mathrm{dx}} &= \sin x \\ v &= -\cos x \\ I &= \int_{x=0}^{x=2\pi} e^{-x/\pi}\sin(x)\mathrm{dx} \\ &= \left[-\cos x e^{-x/\pi}\right]_{x=0}^{x=2\pi} - \frac{1}{\pi}\int_{x=0}^{x=2\pi} e^{-x/\pi}\cos(x)\mathrm{dx} \end{aligned}$$

The last term can be integrated by parts as before (taking u as the exponential),

$$\begin{aligned} I &= \left[-\cos x e^{-x/\pi}\right]_{x=0}^{x=2\pi} - \frac{1}{\pi}\left[\sin x e^{-x/\pi}\right]_{x=0}^{x=2\pi} - \frac{1}{\pi^2}\int_{x=0}^{x=2\pi} e^{-x/\pi}\sin(x)\mathrm{dx} \\ I &= \left[-\cos x e^{-x/\pi}\right]_{x=0}^{x=2\pi} - \frac{1}{\pi}\left[\sin x e^{-x/\pi}\right]_{x=0}^{x=2\pi} - \frac{1}{\pi^2}I \\ &= -\frac{\pi^2}{1+\pi^2}\left[\left(\frac{1}{\pi}\sin x + \cos x\right)e^{-x/\pi}\right]_{x=0}^{x=2\pi} \\ &= -\frac{\pi^2(e^{-2}-1)}{\pi^2+1} \\ &= 0.78509(5d.p.) \end{aligned}$$

9) Calculate the RMS voltage of an AC power supply with $V(t) = V_0 \cos(\omega t)$, between $t = -\pi/\omega$, and $t = \pi/\omega$. Here the constant ω is the frequency of oscillation, and the constant V_0 is the peak voltage.

$$
\begin{aligned}
V_{RMS} &= \sqrt{< V^2 >} \\
&= \sqrt{\frac{1}{b-a}\int_b^a V^2 \mathrm{dt}} \\
V^2 &= V_0^2 \cos^2(\omega t) \\
< V^2 > &= \frac{\omega V_0^2}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} \cos^2 \omega t \mathrm{dt} \\
&= \frac{\omega V_0^2}{4\pi}\int_{-\pi/\omega}^{\pi/\omega} 1 + \cos 2\omega t \mathrm{dt} \\
&= \frac{\omega V_0^2}{4\pi}\left[t + \frac{1}{2\omega}\sin 2\omega t\right]_{-\pi/\omega}^{\pi/\omega} \\
&= \frac{\omega V_0^2}{4\pi}\left[\left(\frac{\pi}{\omega} + 0\right) - \left(-\frac{\pi}{\omega} + 0\right)\right] \\
&= \frac{V_0^2}{2}
\end{aligned}
$$

So the RMS voltage is $V_0/\sqrt{2}$, which is the familiar result we expect.

8 Solutions: Integration III

1) A charged particle produced in e^+e^- annihilation is trapped in the magnetic field of an experiment. The new particle moves with a helical trajectory according to the following equations

$$x = r\cos t,$$
$$y = r\sin t,$$
$$z = ct$$

where the constant r is the radius of the path in the $x - y$ plane, c is a constant corresponding to the rate at which the particle moves along the z axis, and t is time in seconds. Calculate the distance traveled by the charged particle from $t = 0$ to $t = \pi$ seconds. The lectures describes how to calculate arc lengths in two dimensions. The extension to three dimensions is trivial:

$$\begin{aligned}
s &= \int_0^{\pi} \sqrt{\left(\frac{\mathrm{dx}}{\mathrm{dt}}\right)^2 + \left(\frac{\mathrm{dy}}{\mathrm{dt}}\right)^2 + \left(\frac{\mathrm{dz}}{\mathrm{dt}}\right)^2}\mathrm{dt} \\
\frac{\mathrm{dx}}{\mathrm{dt}} &= -r\sin t \\
\frac{\mathrm{dy}}{\mathrm{dt}} &= r\cos t \\
\frac{\mathrm{dz}}{\mathrm{dt}} &= c
\end{aligned}$$

so

$$\begin{aligned}
s &= \int_0^{\pi} \sqrt{r^2(\sin^2 t + \cos^2 t) + c^2}\mathrm{dt} \\
&= \int_0^{\pi} \sqrt{r^2 + c^2}\mathrm{dt} \\
&= \pi\sqrt{r^2 + c}
\end{aligned}$$

2) Calculate the moment of inertia of a strip of uniform density ρ of length L (in the x direction) and height h (along y) about the y axis.

$$\begin{aligned}
\mathrm{dI} &= x^2\mathrm{dm} \\
\mathrm{dI} &= \rho x^2\mathrm{dA} \\
\int \mathrm{dI} &= \int \rho x^2\mathrm{dA} \\
I &= \rho \int_{x=0}^{x=L} \int_{y=0}^{y=h} x^2\mathrm{dydx} \\
&= \rho \int_{x=0}^{x=L} x^2 \left[y\right]_{y=0}^{y=h} \mathrm{dx}
\end{aligned}$$

$$
\begin{aligned}
&= \rho h \int_{x=0}^{x=L} x^2 \mathrm{dx} \\
&= \rho h \left[\frac{x^3}{3} \right]_{x=0}^{x=L} \\
&= \frac{\rho h L^3}{3}
\end{aligned}
$$

3) The lamina defined by the function $y = x\sin(x)$ bounded by the x axis between $x = 0$ and $x = \pi$ is rotated about the y axis to generate a volume. Calculate this volume.

$$
\begin{aligned}
V &= 2\pi \int_0^\pi xy \mathrm{dx} \\
&= 2\pi \int_0^\pi x^2 \sin x \mathrm{dx}
\end{aligned}
$$

so integrate by parts, taking $u = x^2$,

$$
\begin{aligned}
u &= x^2 \\
\frac{\mathrm{du}}{\mathrm{dx}} &= 2x \\
\frac{\mathrm{dv}}{\mathrm{dx}} &= \sin x \\
v &= -\cos x \\
V &= \left[-2\pi x^2 \cos x \right]_0^\pi + 4\pi \int_0^\pi x \cos x \mathrm{dx}
\end{aligned} \tag{3}
$$

and we can do the integral by parts, now taking $u = x$,

$$
\begin{aligned}
u &= x \\
\frac{\mathrm{du}}{\mathrm{dx}} &= 1 \\
\frac{\mathrm{dv}}{\mathrm{dx}} &= \cos x \\
v &= \sin x \\
V &= 2\pi^3 + 4\pi \left[x \sin x \right]_0^\pi - 4\pi \int_0^\pi \sin x \mathrm{dx} \\
&= 2\pi^3 + 0 + 4\pi \left[\cos x \right]_0^\pi \\
&= 2\pi^3 - 8\pi
\end{aligned}
$$

4) The lamina defined by the function $y = ax+1$ bounded by the x axis between $x = 0$ and $x = 1$ is rotated about the x axis to generate a volume. Calculate this volume, and the corresponding centroid positions $\overline{x}$ and $\overline{y}$.

$$\begin{aligned}
\mathrm{dV} &= \pi y^2 \mathrm{dx} \\
V &= \pi \int_0^1 y^2 \mathrm{dx} \\
y^2 &= a^2x^2 + 2ax + 1 \\
V &= \pi \int_0^1 a^2x^2 + 2ax + 1\mathrm{dx} \\
&= \pi \left[\frac{a^2x^3}{3} + ax^2 + x \right]_0^1 \\
&= \pi \left[\frac{a^2}{3} + a + 1 \right]
\end{aligned}$$

By inspection, $\overline{y} = 0$. By considering the moment of the volume about the y axis, we can calculate $\overline{x}$ via

$$\begin{aligned}
\mathrm{dV}\overline{x} &= x(\pi y^2 \mathrm{dx}) \\
V\overline{x} &= \pi \int_0^1 xy^2 \mathrm{dx} \\
&= \pi \int_0^1 a^2x^3 + 2ax^2 + x\mathrm{dx} \\
&= \pi \left[\frac{a^2x^4}{4} + \frac{2ax^3}{3} + \frac{x^2}{2} \right]_0^1 \\
&= \pi \left[\frac{a^2}{4} + \frac{2a}{3} + \frac{1}{2} \right]
\end{aligned}$$

So

$$\overline{x} = \frac{\frac{a^2}{4} + \frac{2a}{3} + \frac{1}{2}}{\frac{a^2}{3} + a + 1}.$$

5) Calculate the surface area generated when the laminar bounded by the x axis, $y = 2x + 1$, $x = 0$, and $x = \pi$ is rotated about the y axis.

$$\begin{aligned}
y &= 2x + 1 \\
\frac{\mathrm{dy}}{\mathrm{dx}} &= 2 \\
\mathrm{dA} &= 2\pi x \mathrm{ds} \\
A &= 2\pi \int x \sqrt{1 + \left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^2} \mathrm{dx}
\end{aligned}$$

$$
\begin{aligned}
&= 2\pi \int_{x=0}^{x=\pi} x\sqrt{1+2^2}\mathrm{dx} \\
&= 2\sqrt{5}\pi \int_{x=0}^{x=\pi} x\mathrm{dx} \\
&= 2\sqrt{5}\pi \left[\frac{x^2}{2}\right]_{x=0}^{x=\pi} \\
&= \sqrt{5}\pi^3
\end{aligned}
$$

6) Calculate the surface area generated by y from Question (4), when rotated about the x axis.

$$
\begin{aligned}
y &= ax+1 \\
\mathrm{dA} &= 2\pi y \mathrm{ds} \\
A &= 2\pi \int_{x=0}^{x=1} y\sqrt{1+\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^2}\mathrm{dx} \\
\frac{\mathrm{dy}}{\mathrm{dx}} &= a \\
A &= 2\pi\sqrt{1+a^2}\int_{x=0}^{x=1}(ax+1)\mathrm{dx} \\
&= 2\pi\sqrt{1+a^2}\left[\frac{ax^2}{2}+x\right]_0^1 \\
&= 2\pi\sqrt{1+a^2}\left[\frac{a}{2}+1\right]
\end{aligned}
$$

7) Calculate the centroid position $(\overline{x}, \overline{y})$ of the volume generated when the laminar defined by $\sin(x)$, the x axis, $x = 0$ and $x = \pi$ is rotated about the y axis.

$$
\begin{aligned}
\mathrm{dV} &= 2\pi xy\mathrm{dx} \\
\overline{x} &= 0\, by\, symmetry \\
\int \overline{y}\mathrm{dM} &= \int y\mathrm{dM} \\
\mathrm{dM} &= \rho\mathrm{dV} \\
\int \overline{y}\mathrm{dV} &= \int y\mathrm{dV} \\
\overline{y} &= \frac{\int y\mathrm{dV}}{\int \mathrm{dV}} \\
&= \frac{\int_0^\pi xy^2\mathrm{dx}}{\int_0^\pi xy\mathrm{dx}}
\end{aligned}
$$

So

$$\begin{aligned}
\int_0^\pi xy\mathrm{dx} &= \int_0^\pi x\sin(x)\mathrm{dx} \\
u &= x \\
\frac{\mathrm{du}}{\mathrm{dx}} &= 1 \\
\frac{\mathrm{dv}}{\mathrm{dx}} &= \sin(x) \\
v &= -\cos(x) \\
\int_0^\pi xy\mathrm{dx} &= [-x\cos(x)]_0^\pi + \int_0^\pi \cos(x)\mathrm{dx} \\
&= \pi
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\pi xy^2\mathrm{dx} &= \int_0^\pi x\sin^2(x)\mathrm{dx} \\
u &= x \\
\frac{\mathrm{du}}{\mathrm{dx}} &= 1 \\
\frac{\mathrm{dv}}{\mathrm{dx}} &= \sin^2(x) \\
v &= \int \sin^2(x)\mathrm{dx}
\end{aligned}$$

we can write $\sin^2(x)$ as $\frac{1}{2}[1-\cos(2x)]$ to calculate v:

$$\begin{aligned}
v &= \frac{1}{2}\int 1-\cos(2x)\mathrm{dx} \\
&= \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right]
\end{aligned}$$

so

$$\begin{aligned}
\int_0^\pi x\sin^2(x)\mathrm{dx} &= \left[\frac{1}{2}\left[x^2 - \frac{x\sin(2x)}{2}\right]\right]_0^\pi - \int_0^\pi \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right]\mathrm{dx} \\
&= \frac{\pi^2}{2} - \frac{1}{2}\left[\frac{x^2}{2} + \frac{\cos(2x)}{4}\right]_0^\pi \\
&= \frac{\pi^2}{2} - \frac{1}{2}\left[\left(\frac{\pi^2}{2} + \frac{1}{4}\right) - \left(0 + \frac{1}{4}\right)\right] \\
&= \frac{\pi^2}{4}
\end{aligned}$$

So $\overline{y} = \frac{\pi^2}{4}\frac{1}{\pi} = \pi/4$.

9 Solutions: Integration IV

1) Calculate the integral $\int_{x=a}^{b}\int_{y=c}^{d} 3x^2 + 2y\,\mathrm{dydx}$.

$$\begin{aligned}
I &= \int_{x=a}^{b}\int_{y=c}^{d} 3x^2 + 2y\,\mathrm{dydx} \\
&= \int_{x=a}^{b} \left[3x^2y + y^2\right]_c^d \mathrm{dx} \\
&= \int_{x=a}^{b} 3x^2(d-c) + (d^2-c^2)\mathrm{dx} \\
&= \left[x^3(d-c) + (d^2-c^2)x\right]_a^b \\
&= (b^3-a^3)(d-c) + (b-a)(d^2-c^2)
\end{aligned}$$

2) Calculate the integral $\int_{r=a}^{b}\int_{\theta=c}^{d}\int_{\phi=e}^{f} r^2(\theta + r\phi)\,\mathrm{d}\phi\mathrm{d}\theta\mathrm{dr}$.

$$\begin{aligned}
I &= \int_{r=a}^{b}\int_{\theta=c}^{d}\int_{\phi=e}^{f} r^2(\theta + r\phi)\,\mathrm{d}\phi\mathrm{d}\theta\mathrm{dr} \\
&= \int_{r=a}^{b}\int_{\theta=c}^{d} \left[r^2\theta\phi + \frac{1}{2}r^3\phi^2\right]_e^f \mathrm{d}\theta\mathrm{dr} \\
&= \int_{r=a}^{b}\int_{\theta=c}^{d} (f-e)r^2\theta + \frac{f^2-e^2}{2}r^3\mathrm{d}\theta\mathrm{dr} \\
&= \int_{r=a}^{b} \frac{(f-e)(d^2-c^2)}{2}r^2 + \frac{(f^2-e^2)(d-c)}{2}r^3\mathrm{dr} \\
&= \frac{(f-e)(d^2-c^2)(b^3-a^3)}{6} + \frac{(f^2-e^2)(d-c)(b^4-a^4)}{8}
\end{aligned}$$

3) The volume element of a cuboid is given by dV = dxdydz. By suitable integration, calculate the volume of a cuboid of dimension $x \times y \times z = 1 \times 2 \times 3$.

$$\begin{aligned}
\mathrm{dV} &= \mathrm{dxdydz} \\
V &= \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3} \mathrm{dzdydx} \\
&= \int_{x=0}^{1}\int_{y=0}^{2} [z]_0^3\mathrm{dydx} \\
&= 3\int_{x=0}^{1}\int_{y=0}^{2} \mathrm{dydx} \\
&= 3\int_{x=0}^{1} [y]_0^2\mathrm{dydx}
\end{aligned}$$

$$
\begin{aligned}
&= 6\int_{x=0}^{1} \mathrm{dydx} \\
&= 6
\end{aligned}
$$

4) By suitable integration, calculate the volume of a sphere of radius R, with volume element $dV = r^2 \sin\theta \mathrm{d}\phi\mathrm{d}\theta\mathrm{dr}$.

$$
\begin{aligned}
\mathrm{dV} &= r^2 \sin\theta \mathrm{drd}\theta\mathrm{d}\phi \\
V &= \int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=-\pi}^{\pi} r^2 \sin\theta \mathrm{d}\phi\mathrm{d}\theta\mathrm{dr} \\
&= \int_{r=0}^{R}\int_{\theta=0}^{\pi} [r^2\phi\sin\theta]_{-\pi}^{\pi}\mathrm{d}\theta\mathrm{dr} \\
&= 2\pi\int_{r=0}^{R}\int_{\theta=0}^{\pi} r^2\sin\theta\mathrm{d}\theta\mathrm{dr} \\
&= 2\pi\int_{r=0}^{R}[-r^2\cos\theta]_0^{\pi}\mathrm{dr} \\
&= 4\pi\int_{r=0}^{R} r^2\mathrm{dr} \\
&= \frac{4\pi r^3}{3}
\end{aligned}
$$

5) A solid is formed by the surface $z = y\sin(\pi y) + x\cos(\pi x)$, the $x - y$ plane and the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$. By suitable integration, determine the volume of the solid in $units^3$.

$$
V = \int_{x=0}^{1}\int_{y=0}^{1} y\sin(\pi y) + x\cos(\pi x)\,\mathrm{dydx}
$$

where the first term is integrated by parts taking $u = y$, so

$$
\begin{aligned}
u &= y \\
\frac{\mathrm{du}}{\mathrm{dy}} &= 1 \\
\frac{\mathrm{dv}}{\mathrm{dy}} &= \sin\pi y \\
v &= \frac{-\cos\pi y}{\pi}, \text{ so} \\
V &= \int_{x=0}^{1}\left[\frac{-y\cos\pi y}{\pi}\right]_{y=0}^{1} - \frac{1}{\pi}\int_{y=0}^{1}\cos\pi y\mathrm{dy} + [xy\cos\pi x]_{y=0}^{1}\,\mathrm{dx} \\
&= \int_{x=0}^{1}\frac{1}{\pi} + \frac{1}{\pi}[\sin\pi y]_{y=0}^{1} + x\cos\pi x\mathrm{dx}
\end{aligned}
$$

$$= \int_{x=0}^{1} \frac{1}{\pi} + x\cos\pi x \mathrm{dx}$$

The second term is integrated by parts, taking $u = x$, so

$$\begin{aligned}
u &= x \\
\frac{\mathrm{du}}{\mathrm{dx}} &= 1 \\
\frac{\mathrm{dv}}{\mathrm{dx}} &= \cos\pi x \\
v &= \frac{\sin\pi x}{\pi}, \text{ so} \\
V &= \frac{1}{\pi} + \left[\frac{x\sin\pi x}{\pi}\right]_{x=0}^{1} - \frac{1}{\pi}\int_{x=0}^{1}\sin\pi x \mathrm{dx} \\
&= \frac{1}{\pi} - \frac{1}{\pi^2}\left[-\cos\pi x\right]_{x=0}^{1} \\
&= \frac{1}{\pi} - \frac{2}{\pi^2}
\end{aligned}$$

6) A solid is enclosed by the two surfaces $z = x+y$ and $z = x^2+3xy$, the planes $x = 0$. $x = 1$, $y = 0$, and $y = 1$. By suitable integration, determine the volume of the solid in $units^3$.

$$\begin{aligned}
V &= \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=x+y}^{z=x^2+3xy} \mathrm{dzdydx}, \\
&= \int_{x=0}^{x=1}\int_{y=0}^{y=1} x^2+3xy-x-y\mathrm{dydx}, \\
&= \int_{x=0}^{x=1}\left[(x^2-x)y + \frac{3xy^2}{2} - \frac{y^2}{2}\right]_{y=0}^{y=1}, \\
&= \int_{x=0}^{x=1} x^2 + \frac{x}{2} - \frac{1}{2}\mathrm{dx}, \\
&= \left[\frac{x^3}{3} + \frac{x^2}{4} - \frac{x}{2}\right]_{x=0}^{x=1}, \\
&= \left[\frac{1}{3} + \frac{1}{4} - \frac{1}{2}\right], \\
&= 1/12.
\end{aligned}$$

7) A solid is enclosed by the surface $z = \pi^2\cos(\pi x)\cos(\pi y)$, the xy plane, and the planes $x = 0$, $x = 0.5$, $y = 0$, and $y = 0.5$. By suitable integration, determine the volume of the solid in $units^3$.

$$\begin{aligned}
V &= \int_0^{0.5}\int_0^{0.5}\int_0^{\pi^2\cos(\pi x)\cos(\pi y)} \mathrm{dzdydx}, \\
&= \pi^2 \int_0^{0.5}\int_0^{0.5} \cos(\pi x)\cos(\pi y)\mathrm{dydx}, \\
&= \pi \int_0^{0.5} \cos(\pi x)\left[\sin(\pi y)\right]_0^{0.5} \mathrm{dx}, \\
&= \pi \int_0^{0.5} \cos(\pi x)\mathrm{dx}, \\
&= \left[\sin(\pi x)\right]_0^{0.5}, \\
&= 1.
\end{aligned}$$

8) A solid is formed by the surface $z = ye^{-x}$, the $x - y$ plane and the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$. By suitable integration, determine the volume of the solid in $units^3$.

$$\begin{aligned}
I &= \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=ye^{-x}} \mathrm{dzdydx} \\
&= \int_{x=0}^{x=1}\int_{y=0}^{y=1} ye^{-x}\mathrm{dydx} \\
&= \int_{x=0}^{x=1}\left[\frac{y^2e^{-x}}{2}\right]_{y=0}^{y=1} \mathrm{dx} \\
&= \frac{1}{2}\int_{x=0}^{x=1} e^{-x}\mathrm{dx} \\
&= \left[-\frac{e^{-x}}{2}\right]_{x=0}^{x=1} \\
&= \frac{1}{2} - \frac{1}{2e}.
\end{aligned}$$

10 Solutions: Fourier Series/Integrals

1) Consider the following function describing a periodic square wave potential

$$\begin{aligned} y(t) &= 1,\, 0 \leq t + nT \leq \frac{T}{2} \\ &= 0, \text{elsewhere.} \end{aligned} \tag{4}$$

Determine the Fourier series corresponding to this function. By inspection $A_0 =$ 0, however one can also show this by performing the following integral

$$\begin{aligned} A_n &= \frac{2}{T}\int_0^{T/2} \cos\left(\frac{2n\pi}{T}t\right) \mathrm{dt} \\ &= \frac{1}{n\pi}\left[\sin\left(\frac{2n\pi}{T}t\right)\right]_0^{T/2} \\ &= \frac{\sin(n\pi) - \sin(0)}{n\pi} \\ &= 0, \forall\, n. \end{aligned}$$

So

$$\begin{aligned} B_n &= \frac{2}{T}\int_0^{T/2} \sin\left(\frac{2n\pi}{T}t\right) \mathrm{dt} \\ &= \frac{1}{n\pi}\left[-\cos\left(\frac{2n\pi}{T}t\right)\right]_0^{T/2} \\ &= \frac{-\cos(n\pi) - \cos(0)}{n\pi} \\ &= \frac{1 - \cos(n\pi)}{n\pi}, \forall\, n. \end{aligned}$$

where $B_n = 0$ for even n, and $B_n = 2/n\pi$ for odd n. The constant term A_0 is given by

$$\begin{aligned} \frac{A_0}{2} &= \frac{1}{T/2 - 0}\int_0^{T/2} \mathrm{dt} \\ &= 1 \end{aligned}$$

and as we have been integrating over half of the period, then we must halve our result (as in the lecture notes) to obtain the correct value of $A_0/2 = 1/2$. So we can now write the Fourier Series as

$$\begin{aligned} y(t) &= \frac{1}{2} + \sum_1^{\infty}\left[\frac{1-\cos(n\pi)}{n\pi}\sin\left(\frac{2n\pi}{T}t\right)\right] \\ &= \frac{1}{2} + \frac{2}{\pi}\sin(2\pi t/T) + \frac{2}{3\pi}\sin(6\pi t/T) + \ldots \end{aligned}$$

Figure 3 shows the first few harmonics of this Fourier Series.

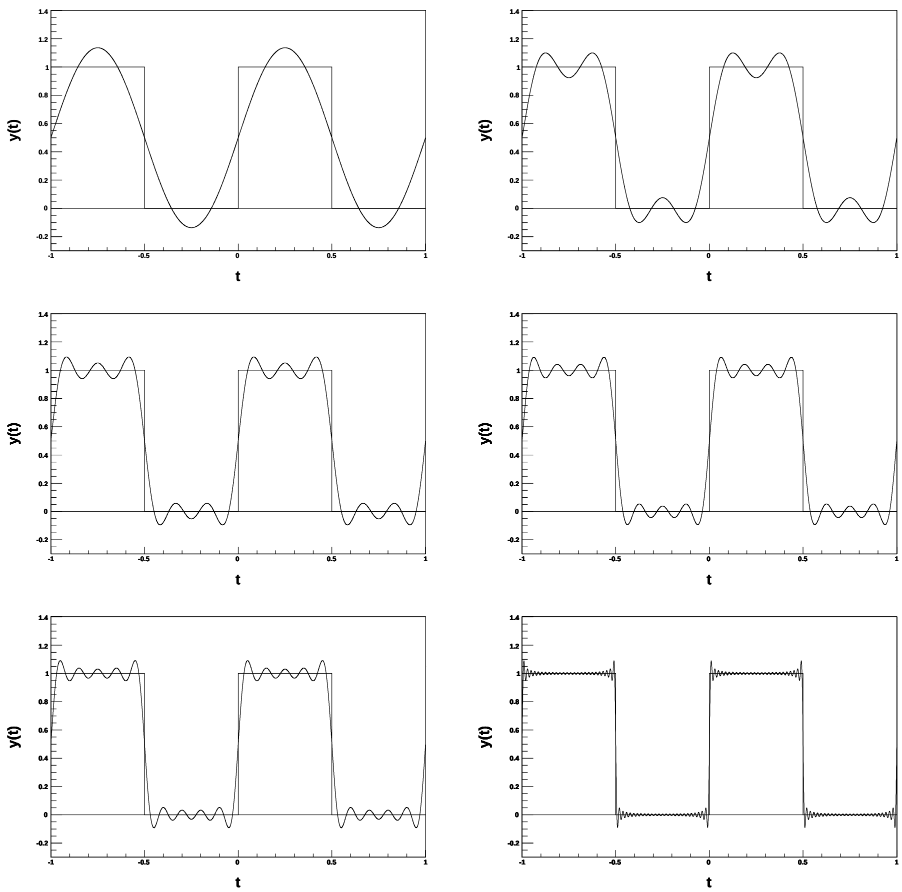

Figure 3: The square wave pulse defined in Question 1 shown over two periods from $t = -1$ to $t = 1$. The corresponding Fourier series for the square wave potential is shown (left to right, top to bottom) including terms up to $n = 1$, 3, 5, 7, 9, and 51. The parameter and T is set to 1 for these plots.

2) Calculate the fourier transforms of the functions (a) $y = \sin(2\pi t/T)$ and (b) $y = e^{-x^2}$. The solutions to these is a standard result related to those given in the lectures:

$$y = \sin(2\pi t/T)$$
$$Y(u) = \frac{1}{2}\left[\delta(u + \frac{1}{T}) - \delta(u - \frac{1}{T})\right]$$

and for

$$
\begin{aligned}
y &= e^{-x^2} \\
Y(u) &= 2\sqrt{\pi}e^{-u^2}
\end{aligned}
$$

3) Calculate $\int_{-\infty}^{\infty} \delta(x-1)f(x)\mathrm{dx}$, where $f(x) = 3x\sin(\pi x/2)$.

$$
\begin{aligned}
\int_{-\infty}^{\infty} \delta(x-1)f(x)\mathrm{dx} &= 3\int_{-\infty}^{\infty} \delta(x-1)x\sin(\pi x/2)\mathrm{dx} \\
&= 3\sin(\pi/2) \\
&= 3
\end{aligned}
$$