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Mathematical Techniques: Revision Notes

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These notes contain the core of the information conveyed in the lectures. They are not a substitute for attending the lectures and none of the examples covered are reproduced here. Worked examples of the techniques described in this note can be found in the tutorial question/solution material provided on the course web site.

34 Fourier Analysis

A Fourier series is built from a sum of periodic components. A benefit of using Fourier series over power series, is that for power series expansions we require our initial function $f(x)$ to be continuous and differentiable over the range in x that we want to be able to use. We can relax these constraints when using Fourier series. Applications of Fourier analysis (the use of Fourier integrals and series) include image processing, voice pattern recognition, and measuring the age of the universe by studying fluctuations in the cosmic microwave background (a remnant of the Big Bang).

35 Fourier Series

Consider a function $y(t)$, which is periodic, such that

$$y(t+T) = y(t), \forall\, t,$$

where T is the period of the function (see Fig. 7). If we need to, we can re-express this condition in terms of the frequency f, or angular frequency ω, by noting the following

$$\omega = \frac{2\pi}{T} = 2\pi f.$$

We can express $y(t)$ in terms of an infinite series of sinusoidal functions; a Fourier series. The general form of a Fourier series is given by Eq. (35.1).

$$\begin{aligned} y(t) &= \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n\cos(n\omega t) + B_n\sin(n\omega t)\right] \\ &= \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{2\pi n}{T}t\right) + B_n\sin\left(\frac{2\pi n}{T}t\right)\right] \end{aligned} \quad (35.1)$$

This is built up of a constant term, plus a sum of even and odd periodic terms. The even periodic terms (symmetric about the origin) are given by the cosines, and the odd periodic terms (asymmetric about the

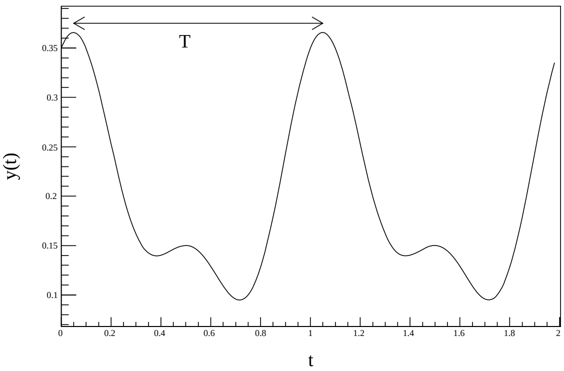

Figure 7: A periodic function $y(t)$. The period of the function T is indicated.

origin) are given by the sines. In order to use a Fourier series to approximate a function, we need to determine the coefficients A_0, A_n and B_n. These are given by

$$\begin{aligned} A_n &= \frac{2}{T}\int_{t_1}^{t_2} y(t)\cos\left(\frac{2\pi n}{T}t\right)dt, \\ B_n &= \frac{2}{T}\int_{t_1}^{t_2} y(t)\sin\left(\frac{2\pi n}{T}t\right)dt, \\ \frac{A_0}{2} &= \frac{1}{2}\frac{2}{T}\int_{t_1}^{t_2} y(t)dt, \\ &= \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} y(t)dt, \\ &= < y(t) >, \end{aligned} \tag{35.2}$$

where $T = t_2 - t_1$, and $< y(t) >$ is the average value of the function over one whole period.

So, for a function $y(t)$ with period T (where $\omega = 2\pi/T$) we can write

$$\begin{aligned} y(t) &= \frac{A_0}{2} + [A_1\cos(\omega t) + B_1\sin(\omega t)] + [A_2\cos(2\omega t) + B_2\sin(2\omega t)] \\ &\quad + [A_3\cos(3\omega t) + B_3\sin(3\omega t)] + \ldots \\ &= \frac{A_0}{2} + \left[A_1\cos\left(\frac{2\pi}{T}t\right) + B_1\sin\left(\frac{2\pi}{T}t\right)\right] \\ &\quad + \left[A_2\cos\left(\frac{4\pi}{T}t\right) + B_2\sin\left(\frac{4\pi}{T}t\right)\right] \\ &\quad + \left[A_3\cos\left(\frac{6\pi}{T}t\right) + B_3\sin\left(\frac{6\pi}{T}t\right)\right] + \ldots \end{aligned}$$

where each term in square brackets is called a *harmonic*. The n^{th} harmonic corresponds to a frequency $f_n = \frac{n\omega}{2\pi} = \frac{n}{T}$.

35.1 Example: Square wave pulses

Consider the example that we have a square wave pulse train as a function of time. This type of situation arises in digital electronic circuits, where the voltage alternates between '1' and '0' (or 'ON' and 'OFF') states. We can express the square wave pulse train as

$$\begin{aligned} y(t) &= h, -\frac{1}{4} \leq t - nT \leq \frac{1}{4} \\ &= 0, \text{elsewhere} \end{aligned} \tag{35.3}$$

where n is any integer. Figure 8 shows the function $y(t)$ over several consecutive periods where the pulse height $h = 1$ and $T = 1$.

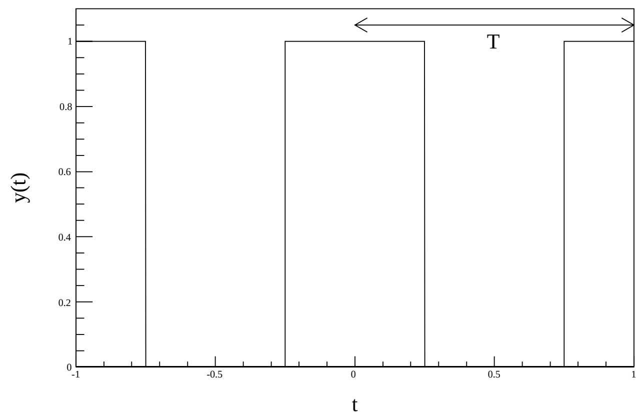

Figure 8: The square wave pulse defined by Eq. (35.3) shown over two periods from $t = -1$ to $t = 1$, where the periodicity of this function $T = 1$, and the pulse height $h = 1$.

The origin is mid-way through a pulse, so $y(t)$ is an even function. With this configuration, we can use the symmetry of the problem to simplify[2] the Fourier series solution for $y(t)$. For this particular example, we only need to integrate from $t = -T/4$ to $t = +T/4$, as the integral outside this range is zero by inspection. Recall from Eq. (35.1) that

$$y(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{2\pi n}{T}t\right) + B_n \sin\left(\frac{2\pi n}{T}t\right) \right]$$

so we can now calculate the coefficients for Eq. (35.3) as follows

$$\begin{aligned} A_0 &= \frac{2}{T}\int_{-T/2}^{T/2} y(t)dt, \\ &= \frac{2}{T}\int_{-T/4}^{T/4} hdt, \\ &= \frac{2}{T}[ht]_{-T/4}^{T/4}, \\ &= h. \end{aligned}$$

[2] With a little practice, you will develop an intuition on sensible choices of the start and end positions of a period of oscillation. This is loosely analogous to choosing the point a about which one expands a Taylor series for a function $f(x)$.

So the average value of $y(t)$ over period is $A_0/2 = h/2$, which is the first coefficient for our Fourier series. Next, we calculate A_n via

$$\begin{aligned}
A_n &= \frac{2}{T}\int_{-T/4}^{T/4} y(t)\cos\left(\frac{2\pi n}{T}t\right)dt, \\
&= \frac{2}{T}\int_{-T/4}^{T/4} h\cos\left(\frac{2\pi n}{T}t\right)dt, \\
&= \frac{2h}{T}\left[\frac{T}{2\pi n}\sin\left(\frac{2\pi n}{T}t\right)\right]_{-T/4}^{T/4} \\
&= \frac{h}{n\pi}\left[\sin\left(\frac{2\pi n}{T}.\frac{T}{4}\right) - \sin\left(\frac{2\pi n}{T}.\frac{-T}{4}\right)\right] \\
&= \frac{h}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{-n\pi}{2}\right)\right] \\
&= \frac{2h}{n\pi}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

If n is odd, then we have $\sin(n\pi/2) = \pm 1$, and A_n is non-zero. If n is even, then $A_n = 0$. Finally we can calculate B_n.

$$\begin{aligned}
B_n &= \frac{2}{T}\int_{-T/4}^{T/4} y(t)\sin\left(\frac{2\pi n}{T}t\right)dt, \\
&= \frac{2}{T}\int_{-T/4}^{T/4} h\sin\left(\frac{2\pi n}{T}t\right)dt, \\
&= \frac{2h}{T}\left[-\frac{T}{2\pi n}\cos\left(\frac{2\pi n}{T}t\right)\right]_{-T/4}^{T/4}, \\
&= \frac{h}{n\pi}\left[-\cos\left(\frac{2\pi n}{T}.\frac{T}{4}\right) + \cos\left(\frac{2\pi n}{T}.\frac{T}{4}\right)\right], \\
&= 0, \forall\, n
\end{aligned}$$

Now we have calculated all coefficients for our Fourier series solution. We can write $y(t)$ as

$$y(t) = \frac{h}{2} + \sum_{n=1}^{\infty}\left[\frac{2h}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{2\pi n}{T}t\right)\right]$$

where we only need to sum over odd values of n. The harmonics can be written down as

$$y(t) = \frac{h}{2} + \frac{2h}{\pi}\cos\left(\frac{2\pi}{T}t\right) - \frac{2h}{3\pi}\cos\left(\frac{6\pi}{T}t\right) + \frac{2h}{5\pi}\cos\left(\frac{10\pi}{T}t\right) + \ldots$$

Figure 9 shows the Fourier series obtained for terms up to n = 1, 3, 5, 7, 9, and 51. As more terms are added, the Fourier series approximation of the square wave function improves.

We can re-write our fourier series solution in terms of ω_0, the angular frequency of the first non-zero harmonic, to look at the frequency content of our function. If we look at the magnitude of our coefficients (in this case, we need only look at A_n for odd n), we see which frequencies are present in our function (See Figure 10).

$$y(t) = \frac{h}{2} + \frac{2h}{\pi}\cos(\omega_0 t) - \frac{2h}{3\pi}\cos(3\omega_0 t) + \frac{2h}{5\pi}\cos(5\omega_0 t) + \ldots \tag{35.4}$$

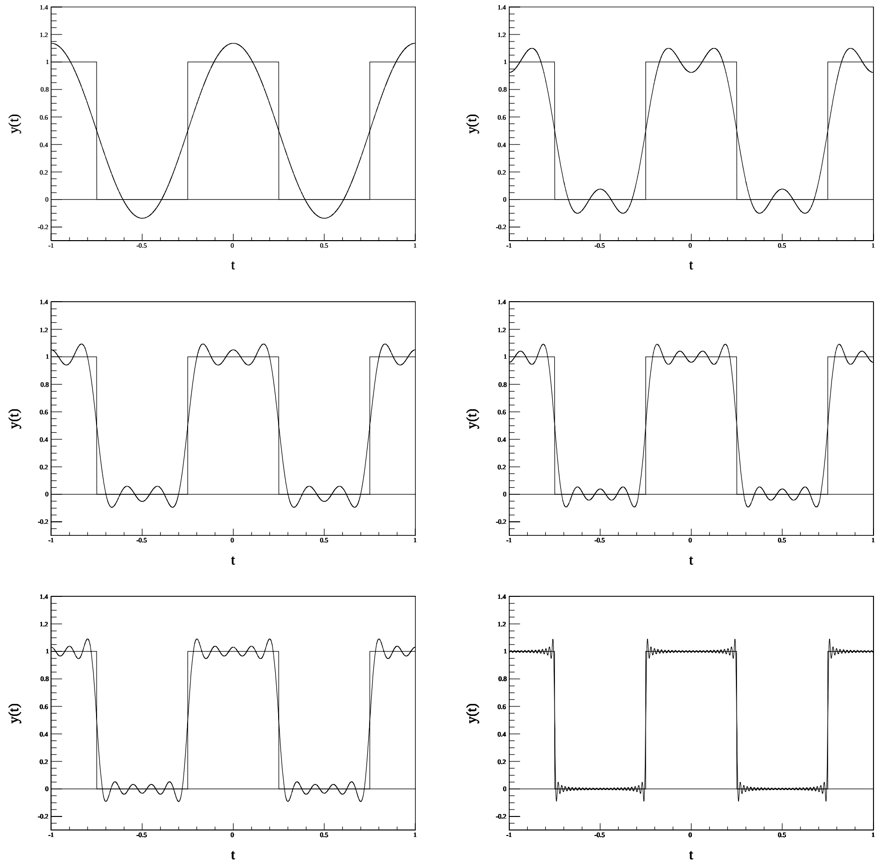

Figure 9: The square wave pulse defined by Eq. (35.3) shown over two periods from $t = -1$ to $t = 1$. The corresponding Fourier Series expansion for the square wave potential is shown (left to right, top to bottom) including terms up to $n = 1$, 3, 5, 7, 9, and 51. The parameters h and T are set to 1 for this Figure.

NOTES:

- We have performed a Fourier Analysis of $y(t)$. This means we have taken our original function, $y(t)$, and expressed this as a function of sine and cosine terms using Eq. (35.1).
- Because we originally chose to have $t = 0$ centered on a pulse with height h, we defined our square wave potential as an even function. The result of this is that our Fourier series does not contain any odd terms. All of the B_n are zero[3]. We could have chosen our origin such that $t = 0$ coincided with the transition between $y(t) = 0$ and $y(t) = h$. If we had done this, then we would have found that our Fourier series would only have contained odd terms (so all A_n would have been zero).
- Constructing a series approximation to our function $y(t)$ by adding successive sine and cosine terms is called synthesis of the Fourier series.
- The solution obtained for $y(t)$ has a parameter for the pulse height, h, and for the period of the function T. We can use our solution for any h and T value we choose. For example, if we set $h = 1$, and $T = 2$ our Fourier series solution for the square wave function is

$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(\pi t) - \frac{2}{3\pi}\cos(3\pi t) + \frac{2}{5\pi}\cos(5\pi t) + \ldots$$

[3] Recall that $\sin(x)$ is an odd function, so we could have guessed that $B_n = 0$ for all n, without having to calculate anything.

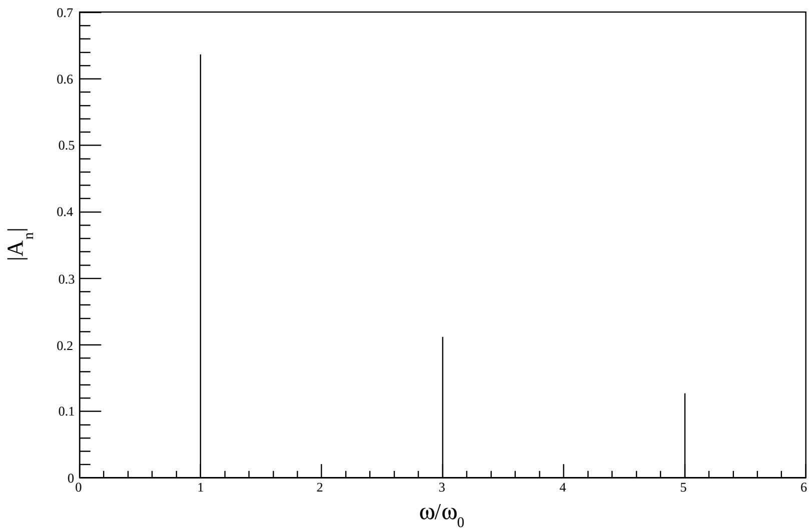

Figure 10: The modulus of the amplitudes with different frequencies contributing to the square wave pulse defined by Eq. (35.3). The corresponding Fourier series for this function is shown in Eq. (35.4)

- We integrated over one full period, but as the solution is periodic, it is valid for all values of t.

35.2 Expressing a Fourier series in terms of exponentials

If we consider the n^{th} harmonic,

$$A_n \cos\left(\frac{2\pi n}{T}t\right) + B_n \sin\left(\frac{2\pi n}{T}t\right),$$

we can use the following relations[4]

$$\begin{aligned}\cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2},\\ \sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{2i},\end{aligned}$$

[4]Recall that $e^{i\theta} = cos\theta + i\sin\theta$.

to rewrite this harmonic as follows (where we let $\theta = \frac{2\pi n}{T} t$):

$$\begin{aligned}
A_n \cos\left(\frac{2\pi n}{T} t\right) + B_n \sin\left(\frac{2\pi n}{T} t\right) &= A_n \cos\theta + B_n \sin\theta \\
&= A_n \frac{e^{i\theta} + e^{-i\theta}}{2} + B_n \frac{e^{i\theta} - e^{-i\theta}}{2i} \\
&= \frac{A_n}{2}(e^{i\theta} + e^{-i\theta}) + \frac{B_n}{2i}(e^{i\theta} - e^{-i\theta}) \\
&= \left(\frac{A_n}{2} + \frac{B_n}{2i}\right) e^{i\theta} + \left(\frac{A_n}{2} - \frac{B_n}{2i}\right) e^{-i\theta} \\
&= C_n^+ e^{i\theta} + C_n^- e^{-i\theta},
\end{aligned}$$

where

$$\begin{aligned}
C_n^{\pm} &= \left(\frac{A_n}{2} \pm \frac{B_n}{2i}\right) \\
&= \left(\frac{A_n}{2} \mp \frac{iB_n}{2}\right)
\end{aligned}$$

So, we can re-write Eq. (35.1) as

$$y(t) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left[C_n^+ e^{i\omega n t} + C_n^- e^{-i\omega n t}\right]$$

as $\theta = n\omega = \frac{2n\pi}{T}$. We can further simplify[5] the Fourier series by writing

$$\begin{aligned}
y(t) &= \sum_{n=-\infty}^{\infty} Y_n e^{i\frac{2\pi n}{T} t}, \\
&= \sum_{n=-\infty}^{\infty} Y_n e^{i\omega n t},
\end{aligned}$$

where

$$\begin{aligned}
Y_n &= \frac{1}{T} \int_{t_1}^{t_2} y(t) e^{-i\frac{2\pi n}{T} t} dt, \\
&= \frac{1}{T} \int_{t_1}^{t_2} y(t) e^{-i\omega n t} dt.
\end{aligned}$$

36 Fourier Integrals

Consider a function which is not necessarily periodic, for example see Figure 11, where we require that $y(t) \to 0$ as $t \to \pm\infty$.

What is the frequency content of $y(t)$? Can we re-write $y(t)$ in terms of the frequency $\nu = \frac{1}{t}$? To do this, we need transform from the time domain t to the frequency domain ν. In general, we can transform from x

[5] Our aim is to obtain integrals for the coefficients that are easier to calculate than the original form.

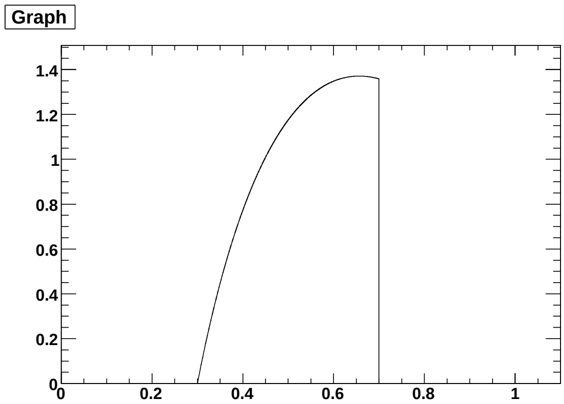

Figure 11: An arbitrary function $y(t)$.

to its reciprocal $u = 1/x$ using:

$$\begin{aligned} Y(u) &= \int_{-\infty}^{\infty} y(x)e^{-i2\pi ux}dx, \\ y(x) &= \int_{-\infty}^{\infty} Y(u)e^{i2\pi ux}du, \end{aligned} \quad (36.1)$$

Note that the exponent in the transformation from x to u has the opposite sign to the exponent in the inverse transformation (from u to x).

36.1 The Dirac δ function

Note that in general, the wider a function is in space, the narrower it is in the reciprocal space. If we take this general rule to the extreme for $f(x) =$ constant, which is equivalent to a 'top hat' function where $L \to \infty$, then we obtain (see Figure 12)

$$\begin{aligned} \lim_{L\to\infty} Y(u) &= \lim_{L\to\infty} \left[\frac{1}{\pi u}\sin(\pi L u)\right], \\ &= \delta(u). \end{aligned}$$

Here $\delta(u)$ is called the Dirac delta function, and is defined as

$$\delta(u) = 0, \text{for } u \neq 0$$
$$\int_{-\infty}^{\infty} \delta(u)du = 1.$$

So the integral over all u of the δ function is unity, and the delta function is non-zero only at the position u. We can create delta functions at different locations, by providing an offset e.g.

- $\delta(u - u_0)$ is a δ function at $u = +u_0$
- $\delta(u + u_0)$ is a δ function at $u = -u_0$

A useful property of the δ function is

$$\int_{-\infty}^{\infty} \delta(u - u_0) f(u) du = f(u = u_0) = f(u_0).$$

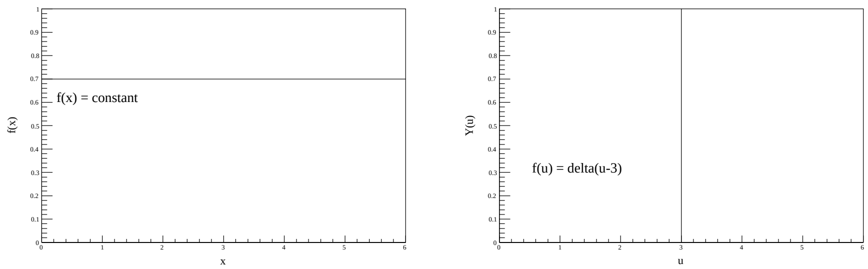

Figure 12: (left) The function $f(x) =$ constant (right) the frequency domain representation $f(x) = \delta(u)$.

36.2 Some useful transforms

Table 4 lists a few standard fourier transforms.

Table 4: Standard fourier transforms $Y(u)$ of functions $y(t)$.

$y(t)$	$Y(u)$
$\cos(2\pi t/T)$	$\frac{1}{2}\left[\delta\left(u + \frac{1}{T}\right) + \delta\left(u - \frac{1}{T}\right)\right]$
$\sin(2\pi t/T)$	$\frac{1}{2}\left[\delta\left(u + \frac{1}{T}\right) - \delta\left(u - \frac{1}{T}\right)\right]$
e^{-x^2/L^2}	$2\sqrt{\pi}e^{-u^2L^2}$
top hat	$\sin(L\pi u)/(\pi u)$