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Mathematical Techniques: Revision Notes

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These notes contain the core of the information conveyed in the lectures. They are not a substitute for attending the lectures and none of the examples covered are reproduced here. Worked examples of the techniques described in this note can be found in the tutorial question/solution material provided on the course web site.

$\sqrt{-1} = i$, where i is an imaginary number. It follows that $i \times i = i^2 = -1$. Imaginary numbers are orthogonal to real numbers.

Complex numbers are composites of real and imaginary numbers:

$$z = a + ib,$$

where a is the real part of the complex number with a magnitude a along the real axis, and ib is the imaginary part, with a magnitude b along the imaginary axis.

Here i denotes an imaginary number, however it is common place to see the symbol j used in engineering circles to denote an imaginary number.

20 Argand Diagrams

We can represent z as a point in the real vs imaginary plane as shown in Figure 6. This type of diagram is called an Argand Diagram.

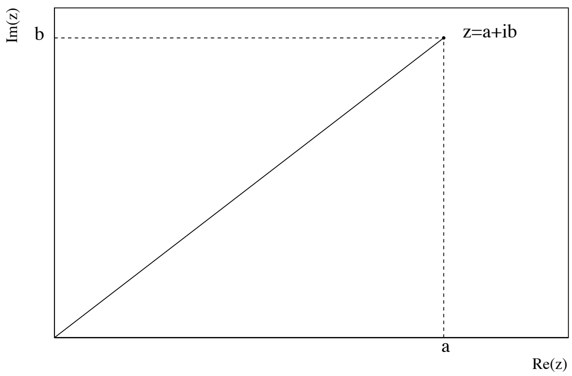

Figure 6: The complex number $z = a + ib$ represented on an Argand Diagram.

z can also be expressed in terms of its distance from the origin r, and the angle subtended between z, the origin and the real axis θ. For $z = a + ib$,

$$r = \sqrt{a^2 + b^2}, \quad (20.1)$$

$$\tan\theta = \frac{b}{a}. \quad (20.2)$$

So

$$z = r(\cos\theta + i\sin\theta).$$

r is the magnitude of the complex number, and θ is the argument of the complex number.

21 Operators

Powers of i

Given that $i = \sqrt{-1}$, it follows that $i^2 = -1$. With this knowledge we are able to evaluate i^n,

$$\begin{aligned} i &= \sqrt{-1}, \\ i^2 &= -1, \\ i^3 &= -i, \end{aligned} \quad (21.1)$$

and so on.

Addition and Subtraction

Given two complex numbers, $z_1 = a + ib$ and $z_2 = c + id$, we can add them together:

$$\begin{aligned} z_3 &= z_1 \pm z_2, \\ &= (a \pm c) + i(b \pm d). \end{aligned}$$

Multiplication

When multiplying two complex numbers $z_1 = a + ib$ and $z_2 = c + id$, together we compute the following product

$$\begin{aligned} z_1 \times z_2 &= (a + ib)(c + id), \\ &= ac + ibc + iad + i^2bd, \\ &= (ac - bd) + i(bc + ad). \end{aligned}$$

Conjugation

The complex conjugate of a complex number z is denoted as z^*. The operation of taking the conjugate of a complex number changes the sign of the imaginary part of z. So for our complex number $z = a + ib$, we find that $z^* = a - ib$. If we multiply a complex number z by its conjugate z^*, we obtain

$$\begin{aligned} zz^* &= (a + ib)(a - id), \\ &= a^2 + b^2. \end{aligned}$$

which is a real number.

Division

When dividing $z_1 = a + ib$ by $z_2 = c + id$, we need to obtain a real number as the denominator. To simplify the problem, we can multiply z_1/z_2 by $(z_2/z_2)^*$ as follows:

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{a+ib}{c+id}, \\
&= \frac{a+ib}{c+id} \times \frac{c-id}{c-id}, \\
&= \frac{(ac+bd)+i(bc-ad)}{c^2+d^2},
\end{aligned}
$$

where the denominator is a real number as desired.

22 The exponential form of a complex number

We obtained the result that $z = a + ib = r(\cos\theta + i \sin\theta)$ in the previous section. We can use the Maclaurin series expansions for $\sin\theta$, $\cos\theta$ and $e^{i\theta}$ to write

$$
\begin{aligned}
z &= r\left[1 - \frac{(\theta)^2}{2!} + \frac{(\theta)^4}{4!} - \frac{(\theta)^6}{6!} + \ldots + i\left(\theta - \frac{(\theta)^3}{3!} + \frac{(\theta)^5}{5!} - \frac{(\theta)^7}{7!} + \ldots\right)\right], \quad (22.1) \\
z &= r\left[1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots\right],
\end{aligned}
$$

where we can associate the terms in square brackets as the Maclaurin series expansion for $e^{i\theta}$.

Now we can revisit how operators act on complex numbers using the exponential form. If we have complex numbers, $z = re^{i\theta}$, $z_1 = r_1 e^{i\theta_1}$, and $z_2 = r_2 e^{i\theta_2}$ then:

$$
\begin{aligned}
zz^* &= r^2, \\
\ln(z) &= \ln(r) + i\theta, \\
z_1 \times z_2 &= r_1 r_2 e^{i(\theta_1+\theta_2)}, \\
\frac{z_1}{z_2} &= \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.
\end{aligned}
$$

It is more straightforward to manipulate the exponential form, as opposed to the cartesian form, when calculating products and quotients of complex numbers.

Sometimes it can be useful to express sine and cosine functions in terms of exponentials, for example when integrating. Given that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, it follows that

$$
\begin{aligned}
\cos(\theta) &= \frac{e^{i\theta} + e^{-i\theta}}{2}, \\
\sin(\theta) &= \frac{e^{i\theta} - e^{-i\theta}}{2i}.
\end{aligned}
$$

23 Powers of a complex number

$$
\begin{aligned}
z &= re^{i\theta}, \\
z^2 &= re^{i\theta} \times re^{i\theta}, \\
&= r^2 e^{i2\theta},
\end{aligned}
$$

and in general, we may write the n^{th} power of z as

$$z^n = r^n e^{in\theta}.$$

This result can be expressed in terms of sine and cosine functions as

$$
\begin{aligned}
z^n &= r^n(\cos\theta + i\sin\theta)^n, \\
&= r^n(\cos(n\theta) + i\sin(n\theta)),
\end{aligned}
$$

which utilizes de Moivre's Theorem[1].

24 Roots of a complex number

$$z^{\frac{1}{2}} = r^{1/2}e^{i\theta/2},$$

but this is not the complete solution, just one part of it. The argument of the square root of a complex number is halved. We need to consider that the complex number is written as $r(\cos\theta + i\sin\theta)$, where unique solutions to the complex number have $0 \leq \theta < 2\pi$. By halving the argument of a complex number we may make room for another unique solution in this range.

For example, if we take the square root of some complex number $z = e^{i\pi/2}$, then $z^{1/2}$ will have an argument with some value between 0 and 2π. One solution to this is simply $z_1 = e^{i\pi/4}$. But if we think in terms of sines and cosines, the complex number z is periodic in terms of its argument, so an equally valid solution is $z_2 = e^{i(\pi/2+2\pi)/2} = e^{i5\pi/4}$. If we consider another solution $z_3 = e^{i(\pi/2+4\pi)/2}$, we see that this occupies the same position on the Argand diagram as z_1 . . . and so on. There are two unique solutions for $z^{1/2}$.

We can generalise the result obtained here to the following rule

$$z^{\frac{1}{n}} = r^{1/n}e^{i(\theta+m2\pi)/n},$$

where $m = 0, 1, 2, \ldots, n-1$ gives the set of n^{th} roots of a complex number. There are n such roots with $0 \leq \arg(z) < 2\pi$.

25 Hyperbolic Functions

The hyperbolic functions are analogues of the trigonometric functions. These functions are useful in a number of branches of mathematics, science, and enginnering. The hyperbolic sine function is given by

[1] i.e. $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

$\sinh(x) = -i\sin(ix)$, and the hyperbolic cosine function is $\cosh(x) = \cos(ix)$. Using the exponential form of a complex number it follows that:

$$\begin{aligned} \sinh(x) &= \frac{1}{2}\left(e^{x} - e^{-x}\right), \\ \cosh(x) &= \frac{1}{2}\left(e^{x} + e^{-x}\right). \end{aligned}$$

The hyperbolic tangent is given by the ratio:

$$\begin{aligned} \tanh(x) &= \frac{\sinh(x)}{\cosh(x)}, \\ &= \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}. \end{aligned}$$

Figure 7 shows distributions of the $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$ functions.

Given these definitions it is possible to determine the derivatives of the hyperbolic functions:

$$\begin{aligned} \frac{d}{dx}[\sinh(x)] &= \cosh(x), \\ \frac{d}{dx}[\cosh(x)] &= \sinh(x), \\ \frac{d}{dx}[\tanh(x)] &= 1 - \tanh^{2}(x). \end{aligned}$$

The corresponding integrals are:

$$\begin{aligned} \int \sinh(x)dx &= \cosh(x) + C, \\ \int \cosh(x)dx &= \sinh(x) + C, \\ \int \tanh(x)dx &= \ln|\cosh(x)| + C. \end{aligned}$$

The inverse hyperbolic functions are defined in an analogous way to the inverse trigonometric functions, and the derivatives and integrals of the inverse hyperbolic functions can be determined using the same methods utilized for the trigonometric counterparts.

25.1 Hyperbola

The equation describing a hyperbola centred on the origin is $(ax)^2 - (bx)^2 = 1$. We can relate this equation to hyperbolic functions by considering a parametric equation in terms of some variable t, where

$$\begin{aligned} x &= \cosh t, \\ y &= \sinh t. \end{aligned}$$

When $t = 0$, then $x = 1$, and $y = 0$. The lower part of the hyperbola for $x > 0$ corresponds to $t < 0$, and the upper part of the curve corresponds to $t > 0$. As the hyperbola is symmetric about the $y-$axis, similar

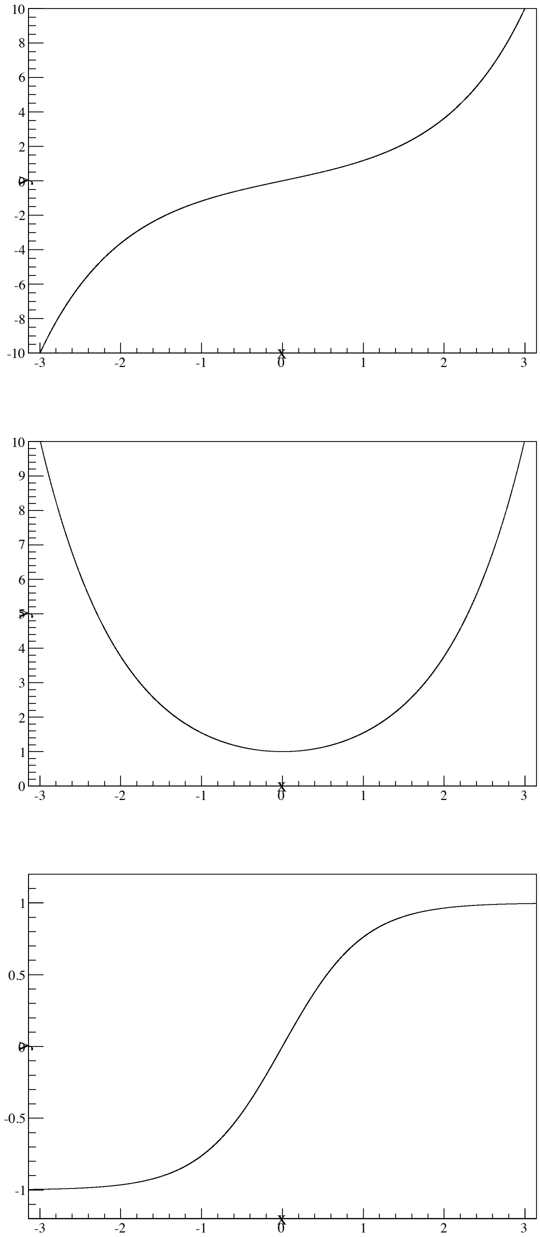

Figure 7: The hyperbolic functions (top) sinh(x), (middle) cosh(x), and (bottom) tanh(x).

behaviour is seen for $x < 0$ where

$$\begin{aligned} x &= -\cosh t, \\ y &= \sinh t. \end{aligned}$$

This is illustrated in Figure 8.

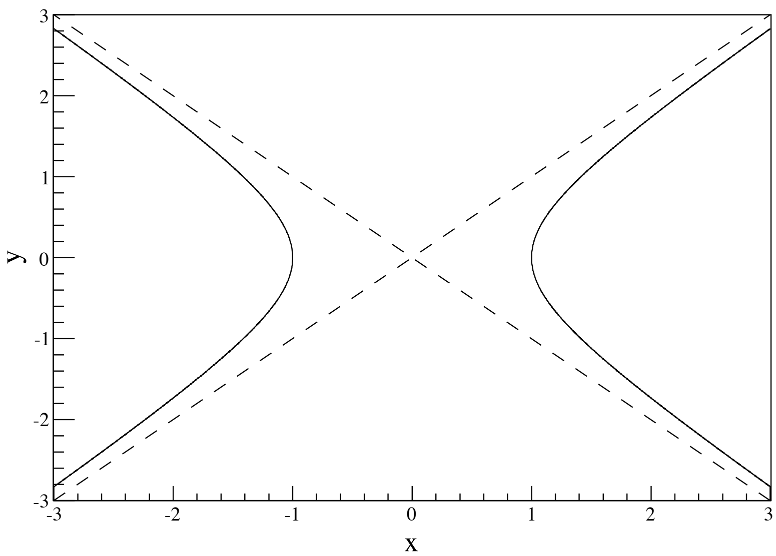

Figure 8: The hyperbola $x^2 - y^2 = 1$.

25.2 Catenary

The use of hyperbolic functions is quite common in engineering and physics applications. One of these is the equation describing a wire linked between two supports. The wire will naturally sag in the middle, and the equation describing the position of the wire between the two supports is called a catenary. The general form of a catenary supported by two vertical posts of the same height is given by:

$$y = a \cosh(x/a) + b.$$

As $\cosh(x = 0) = 1$, the value of $y(x = 0) = a + b$. The parameter a is related to the width of the catenary, where the curve is narrower for a smaller value of a. Figure 9 shows catenaries with different values of a.

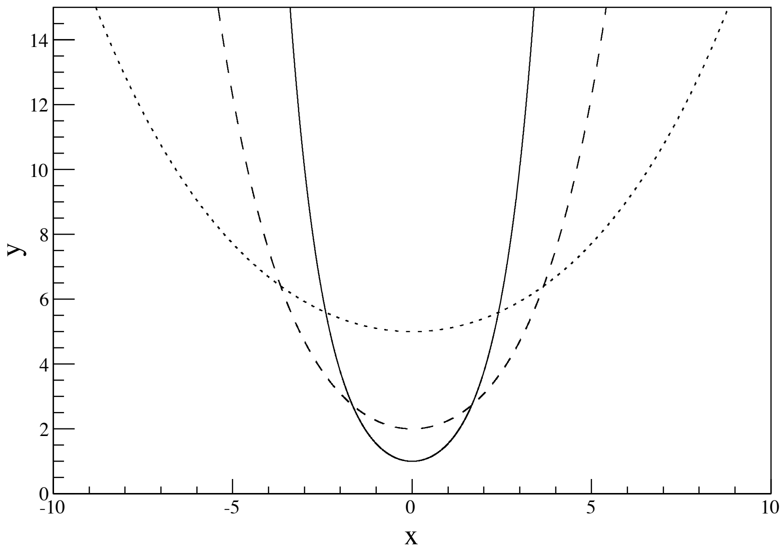

Figure 9: Catenaries for (solid) $a = 1$, (dashed) $a = 2$, and (dotted) $a = 5$.