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Mathematical Techniques: Revision Notes

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These notes contain the core of the information conveyed in the lectures. They are not a substitute for attending the lectures and none of the examples covered are reproduced here. Worked examples of the techniques described in this note can be found in the tutorial question/solution material provided on the course web site.

16 Arithmetic Progression

We can write down the general form of an arithmetic progression (AP) as

$$\sum_{i=0}^{i=n-1} a + id = a + (a + d) + (a + 2d) + \ldots + (a + [n-1]d),$$

where a is the first term of the series, n is the number of terms in the series, and d is the common difference between successive terms of the series. We can calculate the sum S_n of an AP as

$$\begin{aligned} S_n &= a + (a + d) + (a + 2d) + \ldots + (a + [n-1]d), \\ 2S_n &= [a + (a + [n-1]d)] + [(a + d) + (a + [n-2]d)] + \ldots + [a + (a + [n-1]d)], \\ &= n[2a + [n-1]d], \\ S_n &= \frac{n}{2}(2a + [n-1]d). \end{aligned} \tag{16.1}$$

Given an AP with a known values of n, a, and d, we are able to use this result to quickly calculate the sum of the series, even if it has a large number of terms.

17 Geometric Progression

A Geometric Progression (GP) is a series where successive terms differ by some ratio r. We can write down the general form for a GP as

$$\sum_{i=0}^{i=n-1} an^i = a + ar + ar^2 + \ldots + ar^{n-1}.$$

We can calculate S_n for a GP as

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + \ldots + ar^{(n-1)}, \\ rS_n &= ar + ar^2 + ar^3 + \ldots + ar^n, \\ S_n - rS_n &= a - ar^n, \\ S_n &= \frac{a(1 - r^n)}{1 - r}. \end{aligned}$$

So given a GP with known values of n, a and r, we can use this result to quickly calculate the sum of the series, even if it has a large number of terms.

18 Convergence of a Series

Until now we have only considered finite series. It is interesting to ask what happens if we have an infinite number of terms in a series, and can we calculate a meaningful (finite) sum for an infinite series? The answer to this question depends on the infinite series being considered. For example, if you consider the infinite AP

$$\sum_{i=0}^{i=\infty} 1 + id = a + (a + d) + (a + 2d) + \ldots,$$

each successive U_n gets larger, and as the value of n tends to ∞ (we can write this as $n \to \infty$), then we find $S_n \to \infty$. A series with an infinite sum is said not to converge. Conversely a series with a finite sum is said to converge. The infinite AP does not converge as can be seen from Eq. (16.1).

The remainder of this subsection discusses a few of the tests available to check for convergence of a series.

Simple Convergence Test

If the limit $\lim_{n\to\infty} U_n = 0$ for a series, then this series may converge. However this test is not robust and should not be considered conclusive. The usefulness of this test is in its failure, if $\lim_{n\to\infty} U_n \neq 0$ then the series is divergent.

Comparison Test

Comparison test: Test a series against one known to converge. If the n^{th} term in a series is smaller than the n^{th} term is a series known to converge, then the series converges.

D'Alembert's ratio test

D'Alembert's ratio test states that for a series $U_1 + U_2 + \ldots + U_n + \ldots$, look at the limit:

$$R_n = \lim_{n\to\infty} \frac{U_{n+1}}{U_n}.$$

If $R_n < 1$, the series converges; if $R_n > 1$ the series diverges; if $R_n = 1$, one can't tell if the series converges or diverges.

L'Hopital's Rule

L'Hôpital's rule: states

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

This can be useful when trying to determine the limit where one obtains indeterminate results of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

19 Power Series

A power series of some function $f(x)$ of x is one in which the successive terms of the series have increasing powers of x. We can write the general form of a power series as

$$f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \ldots, \tag{19.1}$$

where we need to determine the values of the coefficients $A, B, C, D, E, F, \ldots$

Take the function $f(x) = \sin(x)$ for positive values of x between $x = 0$ and $x = \pi/2$ (See Figure 5). One can see from the figure that the linear term does a poor job of approximating the sine curve by itself. On adding quadratic and cubic terms in x to the power series approximation, the power series approximation becomes a better approximation to the sine curve. If we were to add an infinite number of terms to the power series, then we obtain a solution that exactly matched $f(x) = \sin(x)$. The principle of calculating a power series approximation to an arbitrary function $f(x)$ works in exactly the same way. The aim is to build up an approximation to $f(x)$ by adding terms of increasing order in x to the series, until at some point, the approximation becomes accurate enough for whatever purpose required.

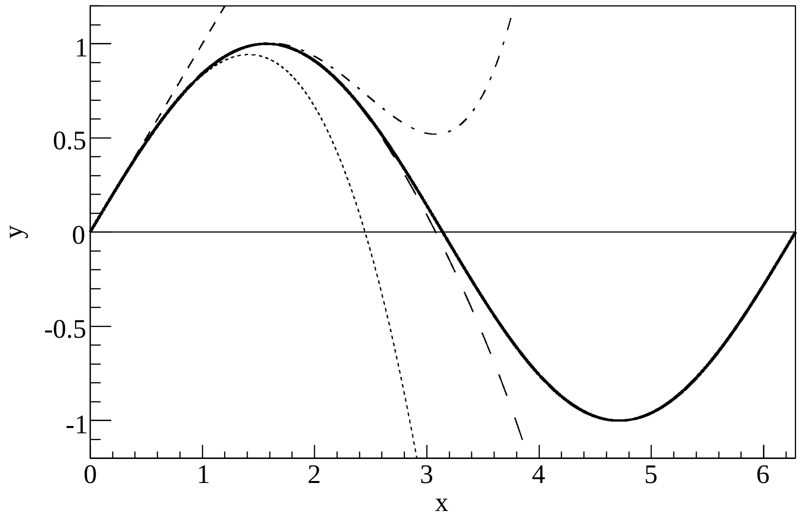

Figure 5: The function $f(x) = \sin(x)$ for positive values of x between $x = 0$ and $x = \pi/2$ with the first few terms of a power series approximation to $f(x)$. The dashed, dotted, dot-dashed and long dashed lines correspond series expansion approximations for $\sin(x)$ using the first one, two, three, and four terms, respectively.

The way we have defined the power series in Eq. (19.1), we have centered the series expansion on $x = 0$. We could equally choose to center the expansion about some point a, in which case the form of the power series would be

$$f(x) = A + B(x-a) + C(x-a)^2 + D(x-a)^3 + E(x-a)^4 + F(x-a)^5 + \ldots, \tag{19.2}$$

where we need to determine the values of the coefficients $A, B, C, D, E, F, \ldots$

19.1 Maclaurin Series

Consider the power series of Eq. (19.1), At $x = 0$ we can see that

$$f(x = 0) = A,$$

Assuming that we are able to differentiate $f(x)$, then we can evaluate

$$f'(x = 0) = B,$$

We can continue this process, as long as we are able to continue to differentiate $f(x)$ to higher orders, to obtain

$$\begin{aligned} f''(x=0) &= 2C, \\ f'''(x=0) &= 2 \times 3D, \\ f''''(x=0) &= 2 \times 3 \times 4E, \end{aligned}$$

etc. Now we can rewrite Eq. (19.1) in terms of the $f(x)$ and its derivatives at $x = 0$ as:

$$f(x) = f_0 + f_0'x + \frac{f_0''}{2!}x^2 + \frac{f_0'''}{3!}x^3 + \frac{f_0''''}{4!}x^4 + \frac{f_0'''''}{5!}x^5 + \ldots, \qquad (19.3)$$

where f_0 denotes $f(x = 0)$, f_0' denotes $f'(x = 0)$ and similarly for the higher derivatives.

19.2 Taylor Series

Consider the power series of Eq. (19.2), At $x = a$ we can see that

$$f(x = a) = A,$$

which gives us the value of the first coefficient. Assuming that we are able to differentiate $f(x)$, then we can evaluate

$$f'(x = a) = B,$$

and from this we obtain the second coefficient. We can continue this process, as long as we can continue to differentiate $f(x)$ to obtain the rest of the coefficients,

$$\begin{aligned} f''(x=a) &= 2C, \\ f'''(x=a) &= 2 \times 3D, \\ f''''(x=a) &= 2 \times 3 \times 4E, \end{aligned}$$

and so on. Given these results, we can rewrite Eq. (19.2) in terms of the $f(x)$ and its derivatives at $x = a$. This gives

$$f_a + f_a'(x-a) + \frac{f_a''}{2!}(x-a)^2 + \frac{f_a'''}{3!}(x-a)^3 + \frac{f_a''''}{4!}(x-a)^4 + \frac{f_a'''''}{5!}(x-a)^5 + \ldots,$$

where f_a denotes $f(x = a)$, f_a' denotes $f'(x = a)$ and similarly for the higher derivatives. If we set $a = 0$ we recover the Maclaurin series.

19.3 Binomial Series

Consider the function $(1 + x)^n$. We can construct a power series approximation to this function by following the same procedure used to obtain the Maclaurin and Taylor series expansions above. We start from

$$(1 + x)^n = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \ldots$$

When $x = 0$, we can see that $A = 1$. If we differentiate both sides of Eq. (19.4) we obtain

$$n(1+x)^{n-1} = B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \ldots,$$

and when $x = 0$ we see that $B = n$. If we again differentiate both sides, we obtain

$$n(n-1)(1+x)^{n-2} = 2C + 2 \times 3Dx + 3 \times 4Ex^2 + 4 \times 5Fx^3 + \ldots,$$

where $C = n(n-1)/2$ when we set $x = 0$. If we continue to do this, we can write down all of the coefficients for the so called binomial series expansion. The result obtained is

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots$$

If $|x| < 1$ the Binomial series expansion converges for all values of n.