Mathematical Techniques I (PHY121)

September 6, 2009

1 Overview

The following details summarise the course (N.B. You are expected to attend lectures and tutorials. Attendance will be monitored.)

Lecturer: Dr. A Bevan, Office on the 5^{th} floor of the department: Rm 503A (*a.j.bevan@qmul.ac.uk*).

Office Hours: I will be available and in my office on Thursdays from 14:30-15:30 for you to come and ask me questions. Any change to this will be announced at the start of the previous lecture.

Pre-requisites: Familiarity with A-Level Mathematics will be assumed.

Assessment: Assessment will be via homework (20% of your final mark) and a summer exam lasting 2h 30m (80% of your mark). Exam rules are available from the departmental secretaries, and more details are available in the Student Handbook.

Lectures: 20

Ancillary teaching: 22 exercise classes

Synopsis: This course covers various techniques of mathematics, mostly calculus, required in the study of the physical sciences: Complex numbers, differentiation, partial differentiation, series, integration, polar co-ordinates and multiple integration. The course structure includes both lectures and self-paced programmed learning, with assessment by course work and regularly spaced examinations.

Reading List: You might find one of the following text books useful while taking this course.

- K. A. Stroud, "Engineering Mathematics", Palgrave MacMillan (2001) 6^{th} Edition. ISBN 978-1-4039-4246-3.
- K. F. Riley, M. P. Hobson, and S. J. Bence, "Mathematical Methods for Physicists and Engineers", Cambridge (2006) 3^{rd} Edition. ISBN 0-521-67971-0.

Course web-site:

`http://www.ph.qmul.ac.uk/~phy121/`

This web page provides details of the course, including these notes, homework questions, tutorial questions, Multiple Choice Questions (a self-test online), and past papers.
NOTE: If an equation given in the lectures, then you should assume that you will be expected to know it, even if it is not included in this study aid! This includes standard integrals and derivatives.
The remainder of this note is a short study aid.

2 General Remarks on Notation

This section serves as a reminder of some of the notation used throughout this course.

Table 1: The list of trigonometric functions, their inverse, and their reciprocal functions.

Name	Trig. Function	Inverse Trig. Function	1/(Trig Function)
sine	$\sin(x)$	$\arcsin(x)$ or $\sin^{-1}(x)$	$\mathrm{cosec}(x)$
cosine	$\cos(x)$	$\arccos(x)$ or $\cos^{-1}(x)$	$\sec(x)$
tangent	$\tan(x)$	$\arctan(x)$ or $\tan^{-1}(x)$	$\cot(x)$
cosecant	$\mathrm{cosec}(x)$	$\mathrm{arccosec}(x)$ or $\mathrm{cosec}^{-1}(x)$	$\sin(x)$
secant	$\sec(x)$	$\mathrm{arcsec}(x)$ or $\sec^{-1}(x)$	$\cos(x)$
cotangent	$\cot(x)$	$\mathrm{arccot}(x)$ or $\cot^{-1}(x)$	$\tan(x)$

3 Differentiation

Notation: The following notation is used throughout the course to indicate the derivative of some function y with respect to some other variable x:

$$\frac{\mathrm{dy}}{\mathrm{dx}},$$
$$y'.$$

and the second derivative is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right) &= \frac{\mathrm{d^2y}}{\mathrm{dx^2}},\\ &= y''.\end{aligned}$$

Similarly for the notation used to denote higher order derivatives. Table 2 lists a number of useful standard derivatives.

- For $y = f(x)$, where $f(x)$ is a complicated function that can be simplified by a substitution $u = g(x)$, so that $y = h(u)$ is easier to differentiate, one can use the chain rule:
$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{du}}\frac{\mathrm{du}}{\mathrm{dx}}. \tag{1}$$
- For $y = f(g(h(x)))$, where $f(x)$ is a complicated function that can be simplified as above. One can use the chain rule extension:
$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{dg}}\frac{\mathrm{dg}}{\mathrm{dh}}\frac{\mathrm{dh}}{\mathrm{dx}}. \tag{2}$$

Table 2: Table of standard derivatives.

$y = f(x)$	$\frac{dy}{dx}$
x^n	nx^{n-1}
e^x	e^x
e^{kx}	ke^{kx}
a^x	$a^x \ln a$
$\ln x$	$\frac{1}{x}$
$log_a x$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\text{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\text{cosec}\, x$	$-\text{cosec}\, x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$1/\cosh^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

- The derivative of a product of two functions of x, u and v is,

$$\frac{\mathrm{d}}{\mathrm{dx}}(uv) = v\frac{\mathrm{du}}{\mathrm{dx}} + u\frac{\mathrm{dv}}{\mathrm{dx}}. \tag{3}$$

- The derivative of a quotient of two functions of x, u and v, is

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{du}}{\mathrm{dx}} - u\frac{\mathrm{dv}}{\mathrm{dx}}}{v^2}. \tag{4}$$

- The radius of curvature of a function y is given by

$$R = \frac{\left[1 + \left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^2\right]^{3/2}}{\frac{\mathrm{d^2y}}{\mathrm{dx^2}}}. \tag{5}$$

- The first and second partial derivatives of a function $z = f(x, y)$ are

$$\frac{\partial \mathrm{z}}{\partial \mathrm{x}}, \frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}^2}, \frac{\partial \mathrm{z}}{\partial \mathrm{y}\partial \mathrm{x}},$$
$$\frac{\partial \mathrm{z}}{\partial \mathrm{y}}, \frac{\partial^2 \mathrm{z}}{\partial \mathrm{y}^2}, \frac{\partial \mathrm{z}}{\partial \mathrm{x}\partial \mathrm{y}}, \tag{6}$$

where

$$\frac{\partial \mathrm{z}}{\partial \mathrm{y}\partial \mathrm{x}} = \frac{\partial \mathrm{z}}{\partial \mathrm{x}\partial \mathrm{y}}. \tag{7}$$

- The total differential, δz, of a function $z = f(x, y)$ is

$$\delta z = \frac{\partial \mathrm{z}}{\partial \mathrm{x}}\delta x + \frac{\partial \mathrm{z}}{\partial \mathrm{y}}\delta y. \tag{8}$$

- Rates of change. If both x and y are functions of time, t, then the total derivative enables us to calculate the rate of change of z with respect to t

$$\frac{\mathrm{dz}}{\mathrm{dt}} = \frac{\partial \mathrm{z}}{\partial \mathrm{x}}\frac{\mathrm{dx}}{\mathrm{dt}} + \frac{\partial \mathrm{z}}{\partial \mathrm{y}}\frac{\mathrm{dy}}{\mathrm{dt}}. \tag{9}$$

- Change of variables. For some function $z = f(x, y)$, where both x and y are functions of two other variables u and v, we can calculate

$$\frac{\partial \mathrm{z}}{\partial \mathrm{u}} = \frac{\partial \mathrm{z}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \mathrm{u}} + \frac{\partial \mathrm{z}}{\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial \mathrm{u}}, \tag{10}$$

$$\frac{\partial \mathrm{z}}{\partial \mathrm{v}} = \frac{\partial \mathrm{z}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \mathrm{v}} + \frac{\partial \mathrm{z}}{\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial \mathrm{v}}, \tag{11}$$

in analogy with the transformation of variables for the rate of change (see above).

4 Integration

The following integration rules are useful to remember, along with the other rules that are covered in lectures, and Table 3 lists a number of standard integral results.

- When integrating

$$\int f'(x)f(x)\mathrm{dx} \tag{12}$$

we can recognize the solution as $f^2(x)/2 + C$.

- When integrating

$$\int \frac{f'(x)}{f(x)}\mathrm{dx} \tag{13}$$

we can recognize the solution as $\ln|f(x)| + C$.

- When integrating a product of two functions by parts, recall

$$\int u\frac{\mathrm{dv}}{\mathrm{dx}}\mathrm{dx} = uv - \int v\frac{\mathrm{du}}{\mathrm{dx}}\mathrm{dx} \tag{14}$$

which is derived from the use of the product rule for differentiation.

Table 3: Table of standard integrals.

Derivative	Integral
$\frac{\mathrm{d}}{\mathrm{dx}}(x^n) = nx^{n-1}$	$\int x^n \mathrm{dx} = \frac{x^{n+1}}{n+1} + C$ [for $n \neq -1$]
$\frac{\mathrm{d}}{\mathrm{dx}}(e^x) = e^x$	$\int e^x \mathrm{dx} = e^x + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(e^{kx}) = ke^{kx}$	$\int e^{kx} \mathrm{dx} = \frac{e^{kx}}{k} + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(a^x) = a^x \ln a$	$\int a^x \mathrm{dx} = \frac{a^x}{\ln a} + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\ln x) = \frac{1}{x}$	$\int \frac{1}{x} \mathrm{dx} = \ln \lvert x \rvert + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\log_a x) = \frac{1}{x \ln a}$	$\int \frac{1}{x \ln a} \mathrm{dx} = \log_a x + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\sin x) = \cos x$	$\int \sin x \mathrm{dx} = -\cos x + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\cos x) = -\sin x$	$\int \cos x \mathrm{dx} = \sin x + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\tan x) = \sec^2 x$	$\int \sec^2 x \mathrm{dx} = \tan x + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\cot x) = -\mathrm{cosec}^2 x$	$\int \mathrm{cosec}^2 x \mathrm{dx} = -\cot x + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\sec x) = \sec x \tan x$	$\int \sec x \tan x \mathrm{dx} = \sec x + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\sinh x) = \cosh x$	$\int \sinh x \mathrm{dx} = \cosh x + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\cosh x) = \sinh x$	$\int \cosh x \mathrm{dx} = \sinh x + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\tanh x) = \frac{1}{\cosh^2 x}$	$\int \frac{1}{\cosh^2 x} \mathrm{dx} = \tanh x + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} \mathrm{dx} = \sin^{-1} x + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{1-x^2}} \mathrm{dx} = \cos^{-1} x + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \mathrm{dx} = \tan^{-1} x + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$	$\int \frac{1}{\sqrt{1+x^2}} \mathrm{dx} = \sinh^{-1} x + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$	$\int \frac{1}{\sqrt{x^2-1}} \mathrm{dx} = \cosh^{-1} x + C$
$\frac{\mathrm{d}}{\mathrm{dx}}(\tanh^{-1} x) = \frac{1}{1-x^2}$	$\int \frac{1}{1-x^2} \mathrm{dx} = \tanh^{-1} x + C$

- Integrals of the form

$$\int \frac{f(x)}{g(x)} \mathrm{dx} \tag{15}$$

where the quotient $\frac{f(x)}{g(x)}$ can be separated into partial fractions, can be re-written in terms of

$$\int \frac{h'(x)}{h(x)} \mathrm{dx} + \int \frac{k'(x)}{k(x)} \mathrm{dx} + \ldots \tag{16}$$

where the solution is of the form $\ln \lvert h(x) \rvert + \ln \lvert k(x) \rvert + \ldots + C$. It is useful to remember the following, when trying to express a quotient in terms of partial fractions:

  – Factors of $(ax + b)$ result in partial fractions of the form $\frac{A}{ax+b}$.

– Factors of $(ax+b)^2$ result in partial fractions of the form $\frac{A}{(ax+b)} + \frac{B}{(ax+b)^2}$.

– Factors of $(ax+b)^3$ result in partial fractions of the form $\frac{A}{(ax+b)} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3}$.

– Factors of ax^2+bx+c result in partial fractions of the form $\frac{Ax+b}{ax^2+bx+c}$.

4.1 Applications of integration

A definite integral of a function $f(x)$, represents the area bounded by the x axis and the curve $f(x)$ between the limits specified in the integral. This is one of many applications of integration. Some useful formulae that are used in this course are listed below.

- The arc-length along a curve ds is just $\sqrt{\mathrm{dx}^2 + \mathrm{dy}^2}$ assuming that one works in the limit where dx tends to zero. Integrating this we obtain

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^2}\,\mathrm{dx} \tag{17}$$

$$s = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{\mathrm{dx}}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{dy}}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \tag{18}$$

  to deal with integrals over x, or some parametric variable θ, respectively.

- The concept of the moment of inertia of a mass dm is used in this course. If a mass dm is rotating about some axis, at a distance r from that axis, then the moment of inertia dI of that mass is defied as $\mathrm{dI} = r^2\mathrm{dm}$. If we integrate over both sides, we obtain

$$I = \int_{m_1}^{m_2} r^2\mathrm{dm} \tag{19}$$

  and if we know the density (mass per unit area) of the mass, we can perform this integral in terms of area, or r when we need to.

- We can calculate the center of gravity $(\overline{x}, \overline{y}, \overline{z})$ of an object by noting that

$$\int \overline{x}\mathrm{dm} = \int x\mathrm{dm} \tag{20}$$

  (and similarly for y and z). As $\overline{x}$ is a constant, and in this course, we only consider objects of uniform density, we can simplify this equation by writing it as

$$\overline{x} = \frac{\int x\mathrm{dA}}{\int \mathrm{dA}} \tag{21}$$

  for plane objects (in 2-dimensions), or alternatively

$$\overline{x} = \frac{\int x\mathrm{dV}}{\int \mathrm{dV}} \tag{22}$$

  for objects that extend into 3-dimensions (with similar equations for y and z). Note that the factors of density drop out as we assumed that the object is of uniform density.

- If we revolve a laminar about the x axis, then the volume element of this object is given by $\mathrm{dV} = \pi y^2 \mathrm{dx}$. So the centroid position $(\overline{x}, \overline{y})$ is given by the equations

$$\overline{y} = 0, \; by\, symmetry \tag{23}$$

$$\overline{x} = \frac{\int x y^2 \mathrm{dx}}{\int y^2 \mathrm{dx}} \tag{24}$$

- If we revolve a laminar about the y axis, then the volume element of this object is given by $\mathrm{dV} = 2\pi x y \mathrm{dx}$. So the centroid position $(\overline{x}, \overline{y})$ is given by the equations

$$\overline{y} = \frac{\int x y^2 \mathrm{dx}}{\int x y \mathrm{dx}} \tag{25}$$

$$\overline{x} = 0, \; by\, symmetry \tag{26}$$

5 Series

- The first n terms of an arithmetic series is $\sum_{i=0}^{n-1} a + id$.
- The first n terms of a geometric series is $\sum_{i=0}^{n-1} ar^i$.
- The Taylor Series expansion $f(x)$ about a point a is

$$f(x) = f_a + f_a'(x-a) + \frac{f_a''}{2!}(x-a)^2 + \frac{f_a'''}{3!}(x-a)^3 + \ldots \tag{27}$$

  where f_a is the function $f(x)$ evaluated at $x = a$, similarly for the derivatives of f.

- The Maclaurin Series expansion for a function $f(x)$ is

$$f(x) = f_0 + f_0' x + \frac{f_0''}{2!}x^2 + \frac{f_0'''}{3!}x^3 + \ldots \tag{28}$$

  where f_0 is the function $f(x)$ evaluated at $x = 0$, similarly for the derivatives of f.

- The Binomial Series expansion $(1+x)^n$ is

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots \tag{29}$$

  The series converges if $|x| < 1$ for any value of n.

- A few useful tests for convergence of a series covered in the lectures are listed below:
  - $\lim_{n\to\infty} U_n = 0$, If this test is satisfied, then the series may converge (not conclusive). However, if this test is not satisfied, then the series is divergent.
  - Comparison test: Test a series against one known to converge. If the n^{th} term in a series is smaller than the n^{th} term is a series known to converge, then the series converges.
  - D'Alembert's ratio test: For a series $U_1 + U_2 + \ldots + U_n + \ldots$, look at the limit:

$$R_n = \lim_{n\to\infty} \frac{U_{n+1}}{U_n}. \tag{30}$$

    If $R_n < 1$, the series converges; if $R_n > 1$ the series diverges; if $R_n = 1$, can't tell if the series converges or diverges.

– L'Hôpital's rule: states

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}. \tag{31}$$

which can be useful when trying to determine the limit where one obtains indeterminate results of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when naively substituting.

6 Complex Numbers

- $i = \sqrt{-1}$.
- If $z = a + ib$, then $Re(z) = a$, and $Im(z) = b$.
- $z = a + ib = re^{i\theta} = r(\cos\theta + i\sin\theta)$, where $r = |z| = \sqrt{a^2 + b^2}$, and $\tan\theta = \frac{b}{a}$.
- The product of two complex numbers z_1 and z_2 is $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$.
- The quotient of two complex numbers z_1 and z_2 is $\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$.
- $z^n = r^n e^{in\theta}$.
- $\sqrt[n]{z} = \sqrt[n]{r}e^{i(\theta+2m\pi)/n}$, where $m = 0, 1, 2, \ldots n-1$ gives the set of unique solutions with an argument between zero and 2π.

Also note the following useful relations between complex exponentials and trigonometric functions

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \tag{32}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \tag{33}$$

7 Fourier Series

A periodic function $f(x)$ can be written as a fourier series of the form

$$y(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{2\pi n}{T}t\right) + B_n\sin\left(\frac{2\pi n}{T}t\right)\right] \tag{34}$$

where the coefficients are

$$A_n = \frac{2}{T}\int_{t_1}^{t_2} y(t)\cos\left(\frac{2\pi n}{T}t\right)\mathrm{dt}, \tag{35}$$

$$B_n = \frac{2}{T}\int_{t_1}^{t_2} y(t)\sin\left(\frac{2\pi n}{T}t\right)\mathrm{dt}, \tag{36}$$

$$\frac{A_0}{2} = \frac{1}{2}\frac{2}{T}\int_{t_1}^{t_2} y(t)\mathrm{dt}, \tag{37}$$

$$= \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} y(t)\mathrm{dt}, \tag{38}$$

$$= < y(t) >, \tag{39}$$

The Fourier series can be re-expressed in terms of the angular frequency ω or frequency f by noting that $\omega = \frac{2\pi}{T} = 2\pi f$.

8 Fourier Integrals

We can transform between space of a function $y(x)$ and the reciprocal space $Y(u)$, where $x = 1/u$ using

$$
\begin{aligned}
Y(u) &= \int_{-\infty}^{\infty} y(x) e^{-i2\pi ux} \mathrm{dx}, \\
y(x) &= \int_{-\infty}^{\infty} Y(u) e^{i2\pi ux} \mathrm{du}. \qquad (40)
\end{aligned}
$$

The Dirac Delta Function is given as

$$
\begin{aligned}
\delta(u - u_0) &= 0, \text{ for } u \neq u_0 \qquad (41) \\
&= \infty, \text{ for } u = u_0 \qquad (42) \\
\int_{-\infty}^{\infty} \delta(u - u_0) \mathrm{du} &= 1. \qquad (43)
\end{aligned}
$$