Chapter 1

Introductory Material

It is assumed that the reader is familiar with the use of algebra, and a number of other techniques that are covered in pre-university mathematics courses. However, some of that material is summarised in this section to serve as a revision aid.

1.1 Polynomial Equations

For a variable continuous variable x valid for any real number between $-\infty$ and $+\infty$ we can write a function $f(x) = y$ in terms of x. The general form of a linear relationship between these two variables is

$$y = ax + b,$$

where both a and b are constants. When $x = 0$, we find that $y = b$, so the constant b corresponds to the value of y when x is zero, alternatively we may say that b is the value of the intercept of the function $y = ax + b$ with the $y-$axis. The constant a is the slope of the function, or how quickly y changes with x.

We can write $y = f(x)$ such that it depends on powers of x. The simplest such equation can be written generally as

$$y = ax^2 + bx + c.$$

We call equations of this type quadratic equations. When $x = 0$ we find that $f(x) = c$, so the constant c again corresponds to the value at which the curve intercepts the $y-$axis. It is possible to determine the points of intersection between the curve and the $x-$axis by finding the roots to $f(x)$ using

$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

As long as the quantity in the surd is positive we will obtain two real results for $x_\pm$, and the function $f(x)$ will cross the $x-$axis twice, once at $x = x_+$, and once at $x = x_-$. If the quantity in the surd is zero, we will obtain a single solution for the crossing point at $x = -b/2a$.

Often it is possible to simplify a quadratic equation by writing

$$y = ax^2 + bx + c.$$

as

$$\begin{aligned} y &= (Ax + B)^2, \\ &= A^2x^2 + 2ABx + B^2. \end{aligned}$$

If a quadratic equation can be rewritten as $y = (Ax + B)^2$, we know that when $x = 0$, $y = B$, and when $y = 0$, $x = -B/A$.

Another possible solution for simplifying a quadratic equation is to reduce it into factors of the form $(Ax+B)$:

$$\begin{aligned} y &= ax^2 + bx + c, \\ &= (Ax + B)(Cx + D), \\ &= ACx^2 + (AD + BC)x + BD. \end{aligned}$$

If it is possible to factorize the problem in this way we can readily identify the values of a, b, and c as

$$\begin{aligned} a &= AC \\ b &= AD + BC \\ c &= BD \end{aligned}$$

If it is possible to factorize a quadratic equation and write it as $y = (Ax + B)(Cx + D)$, then the function crosses the $x-$axis when $x = -B/A$, and $x = -D/C$.

Higher order terms in powers of $a_n x^n$ can be introduced to write down polynomial equations of order n, where a_n is the coefficient of the term corresponding to the n^{th} power of x. For example, if we consider the quadratic equation $y = (x^2 + x + 1)^2$, we can expand the parentheses as follows

$$\begin{aligned} y &= (x^2 + x + 1)^2, \\ &= (x^2 + x + 1) \times (x^2 + x + 1), \end{aligned}$$

where each term in the first parentheses is multiplied by each term in the second parentheses:

$$\begin{aligned} y &= x^2 \times (x^2 + x + 1) + x \times (x^2 + x + 1) + 1 \times (x^2 + x + 1), \\ &= x^4 + x^3 + x^2 + x^3 + x^2 + x + x^2 + x + 1. \end{aligned}$$

This can be simplified by collecting together the terms x^n to give

$$y = x^4 + 2x^3 + 3x^2 + 2x + 1.$$

The procedure of multiplying out each term in in the first set of parentheses with each term in the second set of parentheses can be used to expand out more complicated functions.

1.2 Trigonometric Identities

The sides of a right angled triangle (opposite o, adjacent a, and hypotenuse h) with internal angles 90°, θ_o, and θ_a (See Fig. 1.1) are related via the equation

$$h^2 = o^2 + a^2.$$

The angle opposite the hypotenuse h is 90°. o is the length of the side opposite the angle θ_o, and a is the length of the side adjacent to θ_o (hence opposite θ_a). Trigonometric functions relate the angles to the sides h, o, and a of the triangle:

$$\begin{aligned} \sin(\theta_o) &= \frac{o}{h}, \\ \cos(\theta_a) &= \frac{a}{h}, \\ \tan(\theta_o) &= \frac{o}{a}. \end{aligned}$$

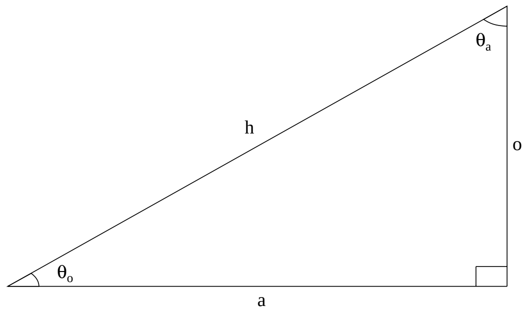

Figure 1.1: A right-angled triangle with sides o, a, and h, and internal angles 90°, θ_o, and θ_a.

For a non-right angled triangle, as depicted in Figure 1.2, we the sides and internal angles are related via the cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos(\theta_a),$$

and via the sine rule:

$$\frac{a}{\sin(\theta_a)} = \frac{b}{\sin(\theta_b)} = \frac{c}{\sin(\theta_c)}.$$

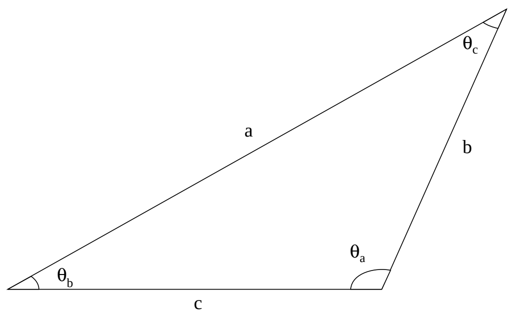

Figure 1.2: A triangle with sides a, b, and c, and internal angles θ_a, θ_b, and θ_c.

The trigonometric functions $\sin(x)$, $\cos(x)$ and $\tan(x)$ are shown in Figure 1.3, and the inverse trigonometric functions are shown in Figure 1.4. Sine and cosine functions are periodic and the values of these functions are repeated every 2π radians. The tangent function is periodic and the value of $\tan(x)$ is repeated every π radians. We can express this mathematically by writing

$$\sin(x) = \sin(x + 2n\pi),$$
$$\cos(x) = \cos(x + 2n\pi),$$
$$\tan(x) = \tan(x + n\pi),$$

where n is any integer. For $x = \pi/2 + n\pi$ $\tan(x)$ is asymptotic, and this corresponds to a collapsed triangle; $\theta_o = \pi/2$ in Figure 1.1. The maximum and minimum values of $\sin(x)$ and $\cos(x)$ are $+1$ and -1.

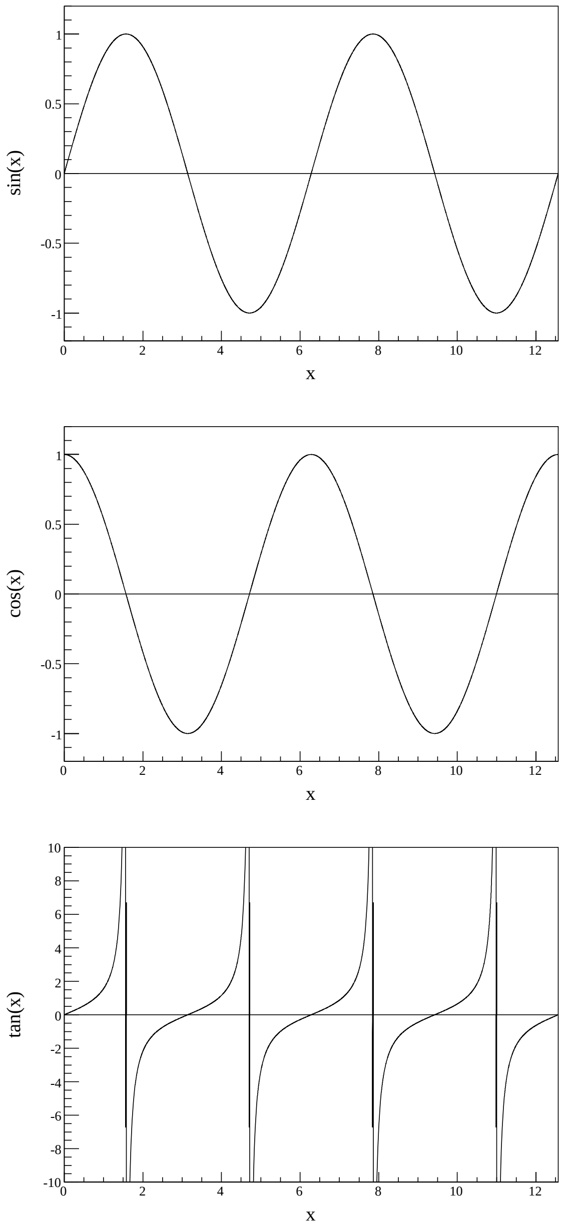

Figure 1.3: The trigonometric functions (top) $\sin(x)$, (middle) $\cos(x)$, and (bottom) $\tan(x)$ for $0 \leq x \leq 4\pi$.

Figure 1.4 shows the inverse trigonometric functions. Unlike the trigonometric functions, the inverse functions are not periodic. $\arcsin(x)$ is the inverse sine function where $-1 \leq x \leq 1$. As sine is periodic, when we compute the $\arcsin(x)$ for a particular value of x we obtain the family of solutions:

$$\arcsin(x) = \theta + 2n\pi, \text{ and } (\pi - \theta) + 2n\pi.$$

Similarly $\arccos(x)$ has the family of solutions:

$$\arccos(x) = \theta + 2n\pi, \text{ and } -\theta + 2n\pi,$$

for $-1 \leq x \leq 1$, and $\arctan(x)$ has the family of solutions:

$$\arctan(x) = \theta + n\pi,$$

for $-\infty \leq x \leq +\infty$. The different ways to express trigonometric functions, their reciprocals, and inverses are listed in table 1.1.

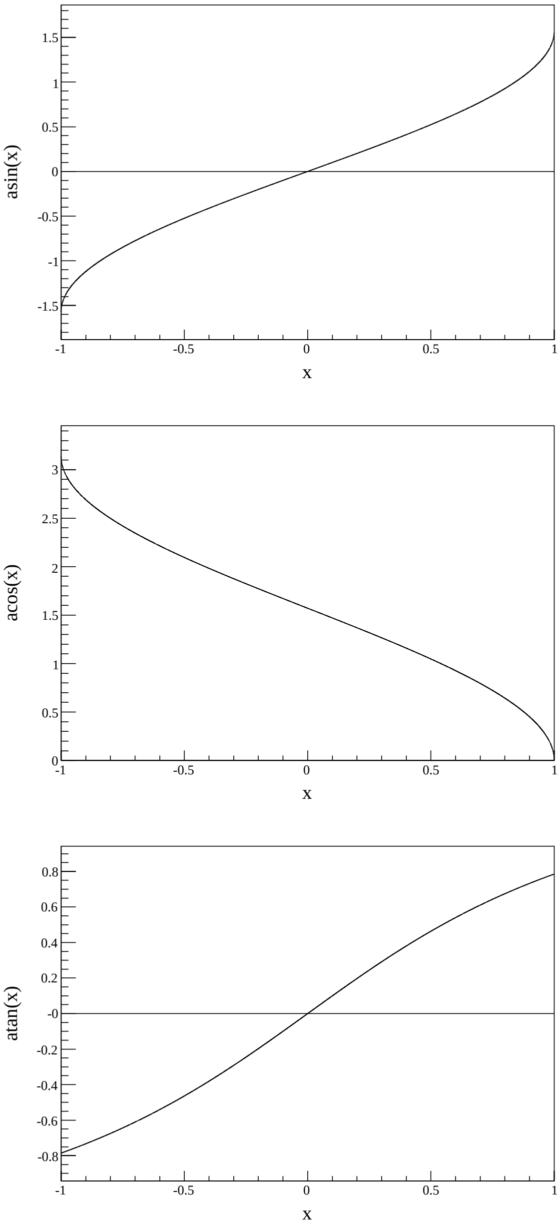

Figure 1.4: The inverse trigonometric functions (top) $\arcsin(x)$, (middle) $\arccos(x)$, and (bottom) $\arctan(x)$ for $0 \leq x \leq 4\pi$.

Table 1.1: The list of trigonometric functions, their inverse, and their reciprocal functions.

Name	Function	Inverse Function	Reciprocal Function
sine	$\sin(x)$	$\arcsin(x)$ or $\sin^{-1}(x)$	$\text{cosec}(x)$
cosine	$\cos(x)$	$\arccos(x)$ or $\cos^{-1}(x)$	$\sec(x)$
tangent	$\tan(x)$	$\arctan(x)$ or $\tan^{-1}(x)$	$\cot(x)$
cosecant	$\text{cosec}(x)$	$\text{arccosec}(x)$ or $\text{cosec}^{-1}(x)$	$\sin(x)$
secant	$\sec(x)$	$\text{arcsec}(x)$ or $\sec^{-1}(x)$	$\cos(x)$
cotangent	$\cot(x)$	$\text{arccot}(x)$ or $\cot^{-1}(x)$	$\tan(x)$

The following trigonometry identities are assumed throughout:

$$\begin{aligned}
\cos^2(x) + \sin^2(x) &= 1, \\
1 + \tan^2(x) &= \sec^2(x), \\
\cot^2(x) + 1 &= \text{cosec}^2(x), \\
\sin^2(x) &= \frac{1}{2}\left[1 - \cos(2x)\right] \\
\cos^2(x) &= \frac{1}{2}\left[1 + \cos(2x)\right] \\
\sin(A + B) &= \sin A \cos B + \cos A \sin B \\
\cos(A + B) &= \cos A \cos B - \sin A \sin B
\end{aligned}$$

where the double angle identities can be obtained in a straightforward way using complex numbers.

1.3 Exponential Function

The general form of an exponential function is written as

$$y = Ae^x,$$

where A is a constant, x is a real number between $-\infty$ and $+\infty$ and e is an irrational quantity called Euler's number, whose value is 2.71828 to 5 decimal places. Figure 1.5 shows the distribution of the function $y = e^x$ in the vicinity of $x = 0$. Exponential functions are the set of functions where the value of rate of change of the function is the same value as the function for a given x (See Section 2.1 for a mathematical discussion of the rate of change of a function). Equations of the type $y = A^x$ are often used when trying to model of population growth.

Products of exponential functions can be re-written as another exponential function

$$e^x e^y = e^{x+y},$$

as is the case for any other product of the same quantity raised to two different powers. Similarly we can write the ratio of two exponential functions as

$$\frac{e^x}{e^y} = e^{x-y}.$$

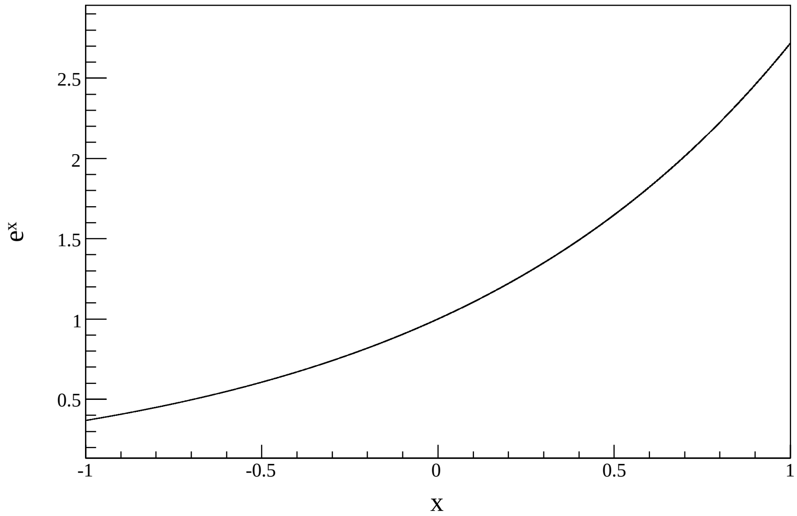

Figure 1.5: The function $y = e^x$.

1.4 Properties of Logarithmic functions

Logarithmic functions are the inverse of exponential functions. We can compute the logarithm of the exponential function $y = e^x$ as

$$\ln(y) = \ln[e^x],$$

as shown in Fig. 1.6. The logarithm denoted by 'ln' is the inverse of the exponential function e, thus

$$\ln(y) = x.$$

We call ln the 'natural logarithm' to highlight the fact that this is the inverse of the function e^x, where e is Euler's number. The natural logarithm of e^{x+y} is $x + y$, so the natural logarithm of the product of two exponentials $e^x e^y$ is $x + y$. It follows that if we have e^x/e^y, then the natural logarithm of this ratio is $x - y$. Logarithms are extremely useful in computations involving products of large numbers, where it is possible to express the problem in terms of the addition of much smaller numbers.

We can write down the logarithm of an exponential function of the form

$$y = N^x,$$

as

$$\log_N(y) = x,$$

where the subscript N denotes the base of the logarithm. Occasionally you may encounter a natural logarithm written as the logarithm of base e, denoted by $\log_e$. The same rules apply for logarithms of some base N as for natural logarithms. For example, if we have the product or quotient of two numbers N^x and N^y, we find that

$$\log_N(N^{x\pm y}) = x \pm y.$$

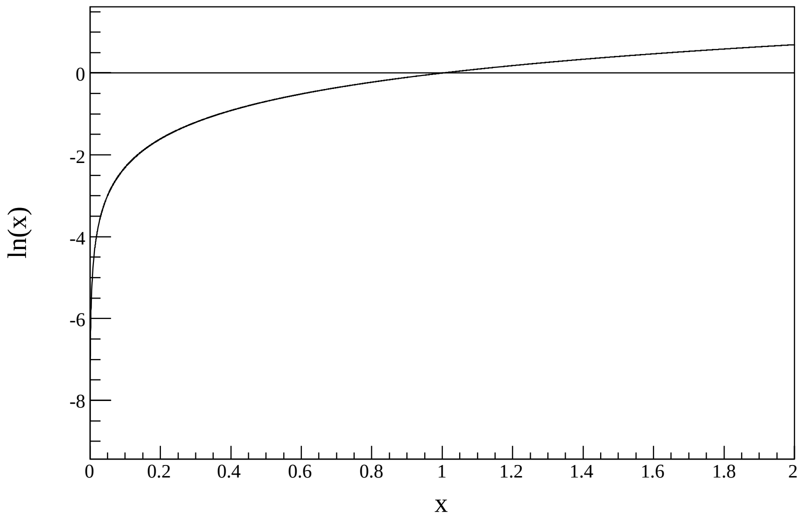

Figure 1.6: The function $y = \ln x$.

Often you will encounter the logarithm denoted by 'log(x)' without specifying the base. In this case the notation implies that one computes the logarithm base 10 of x: $\log_{10}(x)$.

1.5 Partial Fractions

Consider the following function

$$f(x) = \frac{1}{(1-x)(2+x)}$$

It is possible to separate the function into two parts, one with a factor of $(1-x)^{-1}$, the other with a factor of $(2+x)^{-1}$. When we re-write a fraction like this we are separating it into partial fractions. In the above example we may write,

$$\frac{1}{(1-x)(2+x)} = \frac{A}{1-x} + \frac{B}{2+x},$$

where the constants A and B need to be determined. The values of A and B can be determined if we try and recombine the two partial fractions together. This gives

$$\begin{aligned} \frac{A}{1-x} + \frac{B}{2+x} &= \frac{A(2+x) + B(1-x)}{(1-x)(2+x)}, \\ &= \frac{1}{(1-x)(2+x)}. \end{aligned}$$

From this we see that

$$1 = A(2+x) + B(1-x).$$

If we let $x = 1$ we find that $A = 1/3$ and if we let $x = -2$ we find that $B = -1/3$. Using this information we can write our original fraction in terms of partial fractions:

$$\frac{1}{(1-x)(2+x)} = \frac{1}{3(1-x)} - \frac{1}{3(2+x)}.$$

It is possible to use this technique of partial fractions to simplify quotients of more complicated forms. For example, if we have a quotient of the form

$$f(x) = \frac{1}{(ax^2+bx+x)(cx+d)}$$

we can separate this out as

$$\begin{aligned} f(x) &= \frac{A}{(ax^2+bx+x)} + \frac{B}{(cx+d)} \\ &= \frac{A(cx+d)+B(ax^2+bx+x)}{(ax^2+bx+x)(cx+d)}. \end{aligned}$$

As before, we can equate the numerator in the original quotient with the numerator we obtained here, namely

$$A(cx+d) + B(ax^2+bx+x) = 1.$$

All that remains is to determine the values of A and B, which can be done by choosing suitable values for x such that one of the quantities in parentheses '()' is zero, and we can solve for the unknown multiplying the other quantity in parentheses.

1.6 Limiting values

Consider the function $y = 1+x^2$. For any given value of x, we are able to compute a corresponding value for y. If we now consider the behavior of the function in the vicinity of some value of x denoted by x_0, then the value of the function will be given approximately by $y(x = x_0)$. This approximation becomes exact when $x = x_0$, or when x is infinitesimally close to x_0 such that any remaining differences are completely negligible. We often refer to this process as taking a limiting value, or just taking the limit as x tends to x_0. So the limit of function $y = 1 + x^2$ as x tends to x_0 is $1 + x_0^2$. Mathematically we often write this as

$$\lim_{x\to x_0}(y) = 1 + x_0^2.$$

To further illustrate taking limits, we can consider an exponential function, $y = e^x$ as is described in section 1.3. If we take the limiting value of e^x as we allow x to become more and more negative, we find that the value of the function becomes smaller, but never becomes negative. In the extreme case that we allow x to become infinitely negative, then the value of the function becomes vanishingly small, and imperceptible from zero. Now if we consider the case that we let x tend to some finite value, for the sake of illustration we assume that x tends to zero, then the function will tend to e^0. In the limit that x is infinitesimally close, or exactly zero, then in this limit the function has a value of unity, i.e. $e^0 = 1$. If we now consider a third limiting case, where we allow x to become increasingly positive, we see that the value of e^x also becomes increasingly positive. In the extreme limit that we allow x to tend to infinity, then the value of the function also tends to infinity. These results are summarised in the following

$$\lim_{x \to -\infty} e^x = 0,$$
$$\lim_{x \to 0} e^x = 1,$$
$$\lim_{x \to +\infty} e^x = \infty.$$

1.7 Moments

The concept of moments is a fundamental ingredient of classical mechanics. In mechanics the first moment of a force F exerted on a point object is the distance of that point from the origin x multiplied by the force exerted on the object F:

$$1^{st} \text{ moment } = xF.$$

Similarly we can write higher order moments, that are sometimes useful. The n^{th} moment is given by

$$n^{th} \text{ moment } = x^n F.$$