November 27, 2009

Mathematical Techniques: Revision Notes

Dr A. J. Bevan,

These notes contain the core of the information conveyed in the lectures. They are not a substitute for attending the lectures and none of the examples covered are reproduced here. Worked examples of the techniques described in this note can be found in the tutorial question/solution material provided on the course web site.

26 Indefinite Integrals

The process of integration is the reverse of differentiation. If we know the derivative of some function $f(x)$ which is given by $f'(x) = g(x)$, then we know the integral of $g(x)$ with respect to x up to some constant:

$$\frac{d}{dx}[f(x)] = g(x),$$
$$\int g(x)dx = h(x) + C. \tag{26.1}$$

where $f(x) = g(x)$ up to some constant. Here the symbol $\int$ means integrate, and this is followed by the integrand, or function that we want to integrate $g(x)$. The last part of the integral is dx which tells us what variable to integrate over. Table 1 lists standard derivatives, so we can produce a table of standard integrals from this (See Table 3).

27 Definite Integrals

In general we can integrate between two values, or limits, in x, say $x = a$ and $x = b$. If we do this, then we are summing up strips of height $g(x)$ and width δx over this range of values for x. The solution to a definite integral is a number

$$\int_{x=a}^{x=b} g(x)dx = [f(x)]_{x=a}^{x=b},$$
$$= f(b) - f(a),$$

where we have used the shorthand $[f(x)]_{x=a}^{x=b}$ to denote $f(b) - f(a)$. Note that there is no constant of integration for a definite integral.

27.1 Integrals of functions with the form $f'(x)/f(x)$

Consider the following integral

$$\int \frac{f'(x)}{f(x)}dx.$$

Table 3: Table of standard integrals to complement the derivatives given in Table 1.

Derivative	Integral
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ [for $n \neq -1$]
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + C$
$\frac{d}{dx}(e^{kx}) = ke^{kx}$	$\int e^{kx} dx = \frac{e^{kx}}{k} + C$
$\frac{d}{dx}(a^x) = a^x \ln a$	$\int a^x dx = \frac{a^x}{\ln a} + C$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln \lvert x \rvert + C$
$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$	$\int \frac{1}{x \ln a} dx = \log_a x + C$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x dx = -\cos x + C$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x dx = \tan x + C$
$\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$	$\int \mathrm{cosec}^2 x dx = -\cot x + C$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + C$
$\frac{d}{dx}(\mathrm{cosec} x) = -\mathrm{cosec} x \cot x$	$-\int \mathrm{cosec} x \cot x dx = \mathrm{cosec} x + C$
$\frac{d}{dx}(\sinh x) = \cosh x$	$\int \sinh x dx = \cosh x + C$
$\frac{d}{dx}(\cosh x) = \sinh x$	$\int \cosh x dx = \sinh x + C$
$\frac{d}{dx}(\tanh x) = \frac{1}{\cosh^2 x}$	$\int \frac{1}{\cosh^2 x} dx = \tanh x + C$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$
$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x + C$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$	$\int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x + C$
$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$	$\int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + C$
$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$	$\int \frac{1}{1-x^2} dx = \tanh^{-1} x + C$

We can try and simplify the problem by making a substitution for $f(x)$ with some new variable u. The next thing we need to do, is to replace dx in order to integrate in terms of the variable u. So as $u = f(x)$,

$$\frac{du}{dx} = f'(x),$$
$$dx = \frac{du}{f'(x)}.$$

Now we have enough information to re-write the integral

$$\int \frac{f'(x)}{f(x)} dx = \int \frac{f'(x)}{u} dx = \int \frac{1}{u} du.$$

We can see that this integral has a standard form in terms of the variable u, and the solution is just $\ln|u|+C$. We can substitute for u to get the final solution in terms of x, which is

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C.$$

This is a general result and with a little practice one will start to recognise solutions to integrals of this form.

27.2 Integrals of functions with the form $f'(x)f(x)$

Consider the following integral

$$\int f'(x)f(x)dx,$$

where we can simply recognize the integrand as the derivative of the function $f(x)$ multiplied by the function. If we let $u = f(x)$, then

$$\begin{aligned}
\frac{du}{dx} &= f'(x), \\
dx &= \frac{du}{f'(x)}, \\
\int f'(x)f(x)dx &= \int f'(x)u\frac{du}{f'(x)}. \\
&= \int udu, \\
&= \frac{u^2}{2} + C.
\end{aligned}$$

Substituting for u we obtain the solution that

$$\int f'(x)f(x)dx = \frac{[f(x)]^2}{2} + C. \tag{27.1}$$

Example: Consider the following integral

$$I = \int 2x(1+x^2)dx,$$

where we can identify $f(x) = (1+x^2)$, and $f'(x) = 2x$. Using the rule given by Eq. (27.1), we are able to write down the solution as $I = \frac{(1+x^2)^2}{2} + C$. This proposed solution can be checked easily by differentiating:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{(1+x^2)^2}{2} + C\right) &= (1+x^2)\frac{d}{dx}\left(1+x^2\right), \\
&= 2x(1+x^2),
\end{aligned}$$

which is our integrand as required.

28 Integration By Parts

Consider the following integral

$$\int xe^x dx.$$

This is a product of two functions, x and e^x. Individually we know how to differentiate, and how to integrate these two functions. We can use the rule of integration by parts to integrate such a problem. This rule is derived from the product rule for differentiation given in Eq. (3.1) which we can write as

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}.$$

If we integrate both sides of this equation with respect to x we obtain

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx. \tag{28.1}$$

which is the rule for integration by parts. In order to use this rule, we have to identify one function with u, and the second with $\frac{dv}{dx}$. The function u must be differentiable so that we can calculate $\frac{du}{dx}$, and we must be able to integrate $\frac{dv}{dx}$ to calculate v.

When integrating the product $u(x)v(x)$ by parts, if it is possible to isolate $u = x^n$ or e^x, then this should be done. When integrating $ux = x^n e^x$, it is better to take $u = x^n$ (as done for the previous example) in order to solve the integral.

29 Intergration By Parts (contd)

When integrating by parts, sometimes it is possible to end up with the original integral on the RHS of the equation. For example, consider

$$I = \int e^{3x}\sin(x)dx,$$

If we integrate this by parts once we obtain:

$$I = -\cos(x)e^{3x} + 3\int \cos(x)e^{3x}dx$$

which is solved by integrating by parts once more to give

$$\begin{aligned} I &= -\cos(x)e^{3x} + 3\left[\sin(x)e^{3x} - 3\int \cos(x)e^{3x}dx\right] \\ &= -\cos(x)e^{3x} + 3\sin(x)e^{3x} - 9I. \end{aligned}$$

It is easy to determine I from this last step.

30 Integration Using Partial Fractions

We now turn to the set of integrals of the form

$$\int \frac{f(x)}{g(x)}dx,$$

where the quotient can be separated into partial fractions. We can re-write such integrals as the integral of a sum of terms, all of which have the familiar form $f'(x)/f(x)$ (See Section 27.1) and can be solved easily.

If we have an integrand of the form

$$\frac{1}{(Ax+B)(Cx+D)}$$

we can express this as

$$\frac{1}{(Ax+B)(Cx+D)} = \frac{a}{Ax+B} + \frac{b}{Cx+D} \tag{30.1}$$

where we need to determine the values of a and b in order to obtain integrands of the form $f'(x)/f(x)$. To do this we first recombine the right hand side of Eq. (30.1) as follows

$$\frac{a}{Ax+B} + \frac{b}{Ax+B} = \frac{a(Cx+D)+b(Ax+B)}{(Ax+B)(Cx+D)}.$$

From this we see that

$$a(Cx+D)+b(Ax+B) = 1$$

which can be used in order to determine the values of the constants a and b. Hence we can write

$$\begin{aligned}
\int \frac{1}{(Ax+B)(Cx+D)}dx &= \int \frac{a}{Ax+B} + \frac{b}{Cx+D}dx \\
&= \int \frac{a}{Ax+B}dx + \int \frac{b}{Cx+D}dx \\
&= \frac{a}{A}\ln|Ax+B| + \frac{b}{C}\ln|Cx+D| + C
\end{aligned}$$

Note the following

- A linear factor in the denominator $(ax+b)$ gives a partial fraction $A/(ax+b)$.
- A quadratic factor in the denominator $(ax+b)^2$ gives a partial fraction $A/(ax+b)+B/(ax+b)^2$.
- A cubic factor in the denominator $(ax+b)^3$ gives a partial fraction $A/(ax+b)+B/(ax+b)^2+C/(ax+b)^3$.
- Factors of ax^2+bx+c in the denominator give a partial fraction $(Ax+B)/(ax^2+bx+c)$.

31 Notes in integrating trig functions

Use trig identities to simplify the problem; i.e.

$$\begin{aligned}
\cos 2\theta &= \cos^2\theta - \sin^2\theta, \\
&= 2\cos^2\theta - 1, \\
&= 1 - 2\sin^2\theta.
\end{aligned}$$

Other useful identities include

$$\begin{aligned}
2\sin\theta\cos\phi &= \sin(\theta+\phi)+\sin(\theta-\phi), \\
2\cos\theta\cos\phi &= \cos(\theta+\phi)+\cos(\theta-\phi), \\
2\sin\theta\sin\phi &= \cos(\theta-\phi)-\cos(\theta+\phi),
\end{aligned}$$

32 Reduction Formulae

Reduction formulae are formulaic recipes used to solve integrals that would otherwise need a number of iterations before one arrived at a solution. To illustrate how to calculate a reduction formula, consider the integral I_n of $y = x^n e^x$:

$$I_n = \int x^n e^x dx.$$

We are able to integrate y by parts, taking $u = x^n$ and $\frac{dv}{dx} = e^x$. So

$$I_n = x^n e^x - n \int x^{n-1} e^x dx.$$

We can recognise the integral $\int x^{n-1} e^x dx$ as something that is very similar to I_n, and we can call this I_{n-1}. On making this realisation we obtain the reduction formula for I_n:

$$I_n = x^n e^x - n I_{n-1}.$$

We are now able to use this rule recursively in order to calculate the integral for y for any value of n without having to explicitly solve any more integrals. We can use this rule to calculate

$$\begin{aligned} I_1 &= \int x e^x dx. \\ &= x^1 e^x - 1 I_0, \\ &= e^x(x-1) + C, \end{aligned}$$

as required. Reduction formulae can be computed for some integrand $u(x)v(x)$ where $u(x)$ can be expressed as some function raised to the power n.

33 Applications of Integration

33.1 Area Under A Curve

The area of a thin strip of width δx and height y is δA which is given by $\delta A = y\delta x$. In the limit that $\delta x \to 0$ we obtain: $dA = ydx$, so integrating over x for a function between two points a and b is equivalent to summing the area under a curve between a and b:

$$A = \int_a^b y dx.$$

Note that this integral is cumulative, if the function becomes negative, then the area computed will be the sum of the area above and below the $y = 0$ axis. This is not always what you want to compute.

33.2 Parametric functions

Consider the parametric function

$$x = f(\theta), \;\; y = g(\theta)$$

what is

$$I = \int_{\theta=a}^{\theta=b} y dx = \int_{\theta=a}^{\theta=b} g(\theta) dx$$

As

$$\frac{dx}{d\theta} = f'(\theta),$$
$$I = \int_{\theta=a}^{\theta=b} g(\theta) f'(\theta) d\theta$$

33.3 Average value (Mean) of a function

The average value of a function is given by:

$$< y >= \frac{1}{b-a} \int_{x=a}^{x=b} y dx.$$

33.4 RMS of a function

The RMS value of a function is given by

$$RMS(y) = \sqrt{< y^2 >}, = \sqrt{\frac{1}{b-a} \int_{x=a}^{x=b} y^2 dx}.$$

33.5 Arc Length

Consider the arc length Δs corresponding to a change Δx along x, with a corresponding change of Δy in the y direction. Using Pythagoras' theorem we can obtain the approximation

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2,$$
$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2},$$

We can obtain the arc length s between $x = a$ and $x = b$ by taking the limit $\Delta x \to 0$ and integrating both sides of this equation. On integrating the left hand side becomes s, so

$$\begin{aligned} s &= \int \sqrt{(dx)^2 + (dy)^2}, \\ &= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \end{aligned} \tag{33.1}$$

Instead of calculating the arc length by integrating with respect to x, we can equally choose to rearrange Eq. (33.1) in terms of an integration over y

$$s = \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy. \tag{33.2}$$

Similarly, for a parametric equation where $x = x(\theta)$, and $y = y(\theta)$, we can rewrite Eq. (33.1) as an integral over θ

$$s = \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta. \tag{33.3}$$

33.6 Surface Areas

Consider a lamina given by $y(x)$ between $x = a$, and $x = b$. This lamina when revolved about the x-axis produces a surface with a given area. If we consider a thin strip of width dx, the surface area of this strip is the arc length of the strip ds multiplied by the circumference of the surface about the x axis $2\pi y(x)$, i.e.

$$dA = 2\pi y(x)ds.$$

We can integrate both sides of this equation to obtain the surface area

$$\begin{aligned} A &= 2\pi \int_{x=a}^{x=b} y(x)ds, \\ &= 2\pi \int_{x=a}^{x=b} y(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx, \end{aligned}$$

As in Section 33.5, we can rewrite this integral in terms of y, or some parametric variable θ if it helps simplify the problem.

It is also possible to revolve the lamina $y(x)$ about the y axis, instead of the x axis. If this is done, then the surface area of a thin strip of the lamina is given by

$$dA = 2\pi x ds.$$

So the area generated is

$$\begin{aligned} A &= 2\pi \int\limits_{x=a}^{x=b} x ds, \\ &= 2\pi \int\limits_{x=a}^{x=b} x\sqrt{1+\left(\frac{dy}{dx}\right)^2} dx. \end{aligned}$$

33.7 Volumes of Revolution

Consider the lamina in Section 33.6. When this is revolved about the x axis it generates a volume with elemental area

$$dV = \pi y^2 dx.$$

If we integrate the lamina we obtain a volume

$$V = \pi \int\limits_{x=a}^{x=b} y^2 dx.$$

If we revolved the lamina about the y axis instead, the elemental area of volume is

$$dV = 2\pi xy dx,$$

so the volume generated is given by

$$V = \int\limits_{x=a}^{x=b} 2\pi xy dx.$$

If it was more convinient to do so, we could have equally chosen the volume element $dV = \pi x^2 dy$ and perform the integral over y in order to obtain the volume V.

33.8 Centroids of Volumes of Revolution

The point of center of gravity of an object is the point such that there is an equal mass above and below that point. The centroid of a massless object or shape can be computed an an analogous way as the center of gravity. We sum up the moments about an axis of an element of the shape, and integrate over the whole shape in order to compute the centroid positions. It can useful to consider symmetry when computing centroids of volumes.

We can determine the point of center of gravity of an object of mass M, which in one dimension is given by $\overline{x}$ as

$$\int \overline{x} dM = \int x dM, \tag{33.4}$$

as $\overline{x}$ is a constant, we can take this out of the integral and rearrange to give

$$\overline{x} = \frac{\int x dM}{\int dM}, \tag{33.5}$$

where the integrals are over the full extent of the object. This integral can be re-written in terms of the volume by noting that $dM = \rho dx$, where ρ is the density of the object. For a constant density throughout the object we obtain

$$\overline{x} = \frac{\int x dx}{\int dx}. \tag{33.6}$$

If we consider an extended object in three dimensions we can replace x with the vector $\underline{r} = (x, y, z)$ where

$$\overline{\underline{r}} = \frac{\int r d\underline{r}}{\int d\underline{r}}, \tag{33.7}$$

which can be written as three separate equations:

$$\overline{x} = \frac{\int x dx}{\int dx}, \qquad \overline{y} = \frac{\int y dy}{\int dy}, \qquad \overline{z} = \frac{\int z dz}{\int dz}. \tag{33.8}$$

If we integrate massless objects in order to find the mid point, that would correspond to the center of mass in a massive object, we call that point the centroid. We can use Eq. 33.7 to compute the centroid of the thin strip.

$$\begin{aligned}
\overline{x} &= \frac{\int\limits_{x=x_0}^{x=x_0+\Delta x} x dx}{\int_{x=x_0}^{x=x_0+\Delta x} dx}, \\
&= \frac{[x^2/2]_{x=x_0}^{x=x_0+\Delta x}}{[x]_{x=x_0}^{x=x_0+\Delta x}}, \\
&= x_0 + \frac{\Delta x}{2}.
\end{aligned}$$

Similarly for y we find

$$\begin{aligned}
\overline{y} &= \frac{\int\limits_{y=0}^{y=y_0} y dy}{\int_{y=0}^{y=y_0} dy}, \\
&= \frac{[y^2/2]_{y=0}^{y=y_0}}{[y]_{y=0}^{y=y_0}}, \\
&= \frac{y_0}{2}.
\end{aligned}$$

So the centroid position of the thin strip is $(\overline{x}, \overline{y}) = (x_0 + \Delta x, y_0/2)$. If we consider the limit that the strip width Δx tends to zero, then the centroid is just $(x_0, y_0/2)$.

- If we revolve a laminar about the x axis, then the volume element of this object is given by $dV = \pi y^2 dx$. So the centroid position $(\overline{x}, \overline{y})$ is given by the equations

$$\overline{y} = 0, \, by \, symmetry \tag{33.9}$$

$$\overline{x} = \frac{\int xy^2 dx}{\int y^2 dx} \tag{33.10}$$

- If we revolve a laminar about the y axis, then the volume element of this object is given by $dV = 2\pi xy dx$. So the centroid position $(\overline{x}, \overline{y})$ is given by the equations

$$\overline{y} = \frac{\int xy^2 dx}{\int xy dx} \tag{33.11}$$

$$\overline{x} = 0, \, by \, symmetry \tag{33.12}$$

33.9 Moments of Inertia

The moment of inertia dI of a mass element dm rotating about an axis and a distance r from the axis is given by

$$dI = r^2 dm. \tag{33.13}$$

We can compute the moment of inertia I of an extended mass by integrating both sides of Eq. (33.13) to obtain

$$I = \int_m r^2 dm. \tag{33.14}$$

34 Multiple integrals

The previous lectures have started to deal with integrating over more than one dimension (multiple integration). These lectures cover aspects of multiple integration in more detail.

When we considered differentiating a function of two or more variables x, y, z, ... we noted that these variables are orthogonal (or independent). Using this fact it is possible to differentiate a function with respect to one of the variables, keeping all of the rest constant. The same approach can be taken with integrating functions of more than one variable. If we consider $z = f(x, y)$, where x and y are independent, then we can write the integral of this function over x and y as

$$I = \int_y \int_x f(x, y) dx dy.$$

When we write down a multiple integral, the outermost $\int$ sign is paired with the outermost variable to integrate over (dy in this case). Subsequent pairings occur, like layers of an onion, until the innermost layer is reached (the integral over dx in this case).

We are free to integrate with respect to one of the variables, for example x to obtain an intermediate step

$$I = \int_y g(x, y) dy,$$

where in general the functional form of the integrand g still depends on both x and y for an indefinite integral. If we are performing a definite integral, then the functional form of g will be independent of the variable(s) that we have integrated over. The final step in solving this problem involves a second integration, this time over the other variable,

$$I = h(x, y) + C,$$

where the general solution of an indefinite integral is also a function of both x and y.

Example: Integrate the following

$$\begin{aligned} I &= \int\int x\sin(y)dxdy, \\ &= \int \frac{x^2\sin(y)}{2}dy, \\ &= -\frac{x^2\cos(y)}{2} + C. \end{aligned}$$

If the original integral was a definite integral, then the final solution would be a number, whose numerical value depends on the functional form of $f(x, y)$, and on the limits of integration over both x and y. When we integrate a function of one variable between two limits, we calculate an area. So when we integrate a function of two variables, we are calculating a volume.

Example: Calculate the volume bound by the function $f(x, y) = xy^2$ and the $x - y$ plane between $x = 0$, $x = 1$, $y = 0$, and $y = 5$. The integral I is

$$\begin{aligned} I &= \int\limits_{y=0}^{y=5}\int\limits_{x=0}^{x=1} xy^2dxdy, \\ &= \int\limits_{y=0}^{y=5}\left[\frac{x^2y^2}{2}\right]_{x=0}^{x=1}dy, \\ &= \int\limits_{y=0}^{y=5}\frac{y^2}{2}dy, \\ &= \left[\frac{y^3}{6}\right]_{y=0}^{y=5}, \\ &= \frac{125}{6}. \end{aligned}$$

As we have a convention for pairing up each $\int$ with the variable to integrate over, we don't have to explicitly write down the variable when writing down the integration limits.

We can continue to solve integrals with increasing numbers of dimensions using the rules outlined above. For example, we can consider trying to solve the triple integral

$$I = \int\limits_z\int\limits_y\int\limits_x f(x, y, z)dxdydz,$$

where the general solution will also be a function of the variables x, y, and z, up to the integration constant. If we solve a triple integral that is definite, we will obtain a numerical solution.

34.1 Calculating the volume bounded by two surfaces

Consider the problem where we have two surfaces, defined by z_1 and z_2 where

$$\begin{aligned} z_1(x, y) &= f(x, y), \\ z_2(x, y) &= g(x, y). \end{aligned} \tag{34.1}$$

We can calculate the volume bounded between these two surfaces and the planes defined by $x = a$, $x = b$, $y = c$, and $y = d$ by computing the following integral

$$\begin{aligned} V &= \int_{z=f(x,y)}^{g(x,y)} \int_{y=c}^{y=b} \int_{x=a}^{x=b} dxdydz, \\ &= \int_{z=f(x,y)}^{g(x,y)} \int_{y=c}^{y=b} \int_{x=a}^{x=b} g(x, y) - f(x, y)dxdy. \end{aligned}$$

34.2 Integration in spherical polar coordinates

When we integrate in cartesian coordinates, we compute a volume element $dxdydx$, and sum up over all space to compute the integral of all such volume elements for the problem at hand. If we consider spherical polar coordinates, where $0 \le r \le R$, $0 \le \phi \le 2\pi$, and $0 \le \theta \le \pi$, then the volume element in that coordinate system basis is $r^2 \sin\theta dr d\theta d\phi$.

NOTE: This is useful for solid state physics, crystallography as well as classical and quantum mechanical solutions to problems with spherically symmetric potential energies.