Chapter 6

EIGENVALUES AND EIGENVECTORS

6.1 Motivation

We motivate the chapter on eigenvalues by discussing the equation

$$ax^2 + 2hxy + by^2 = c,$$

where not all of a, h, b are zero. The expression $ax^2 + 2hxy + by^2$ is called a *quadratic form* in x and y and we have the identity

$$ax^2 + 2hxy + by^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = X^t AX,$$

where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$. A is called the matrix of the quadratic form.

We now rotate the x, y axes anticlockwise through θ radians to new x_1, y_1 axes. The equations describing the rotation of axes are derived as follows:

Let P have coordinates (x, y) relative to the x, y axes and coordinates (x_1, y_1) relative to the x_1, y_1 axes. Then referring to Figure 6.1:

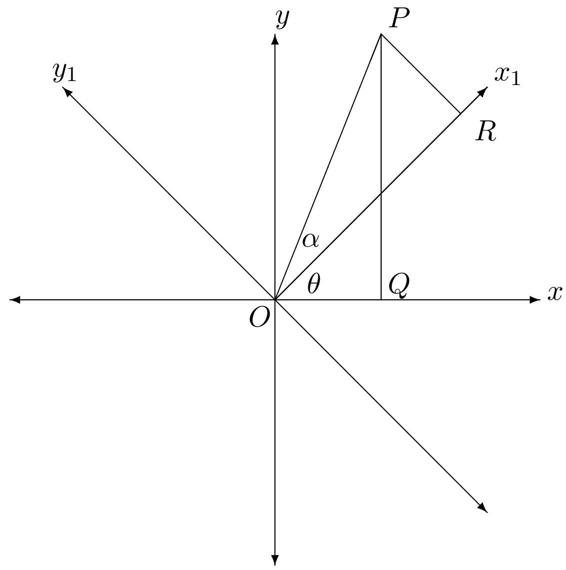

Figure 6.1: Rotating the axes.

$$\begin{aligned} x &= OQ = OP\cos\,(\theta+\alpha) \\ &= OP(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \\ &= (OP\cos\alpha)\cos\theta - (OP\sin\alpha)\sin\theta \\ &= OR\cos\theta - PR\sin\theta \\ &= x_1\cos\theta - y_1\sin\theta. \end{aligned}$$

Similarly $y = x_1\sin\theta + y_1\cos\theta$.

We can combine these transformation equations into the single matrix equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix},$$

or $X = PY$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $Y = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. We note that the columns of P give the directions of the positive x_1 and y_1 axes. Also P is an orthogonal matrix – we have $PP^t = I_2$ and so $P^{-1} = P^t$. The matrix P has the special property that $\det P = 1$.

A matrix of the type $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is called a *rotation* matrix. We shall show soon that any 2×2 real orthogonal matrix with determinant equal to 1 is a rotation matrix.

We can also solve for the new coordinates in terms of the old ones:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = Y = P^t X = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

so $x_1 = x\cos\theta + y\sin\theta$ and $y_1 = -x\sin\theta + y\cos\theta$. Then

$$X^t AX = (PY)^t A(PY) = Y^t(P^t AP)Y.$$

Now suppose, as we later show, that it is possible to choose an angle θ so that $P^t AP$ is a diagonal matrix, say $\text{diag}(\lambda_1, \lambda_2)$. Then

$$X^t AX = \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \lambda_1 x_1^2 + \lambda_2 y_1^2 \qquad (6.1)$$

and relative to the new axes, the equation $ax^2 + 2hxy + by^2 = c$ becomes $\lambda_1 x_1^2 + \lambda_2 y_1^2 = c$, which is quite easy to sketch. This curve is symmetrical about the x_1 and y_1 axes, with P_1 and P_2, the respective columns of P, giving the directions of the axes of symmetry.

Also it can be verified that P_1 and P_2 satisfy the equations

$$AP_1 = \lambda_1 P_1 \text{ and } AP_2 = \lambda_2 P_2.$$

These equations force a restriction on λ_1 and λ_2. For if $P_1 = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$, the first equation becomes

$$\begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \text{ or } \begin{bmatrix} a - \lambda_1 & h \\ h & b - \lambda_1 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence we are dealing with a homogeneous system of two linear equations in two unknowns, having a non–trivial solution (u_1, v_1). Hence

$$\begin{vmatrix} a - \lambda_1 & h \\ h & b - \lambda_1 \end{vmatrix} = 0.$$

Similarly, λ_2 satisfies the same equation. In expanded form, λ_1 and λ_2 satisfy

$$\lambda^2 - (a + b)\lambda + ab - h^2 = 0.$$

This equation has real roots

$$\lambda = \frac{a + b \pm \sqrt{(a + b)^2 - 4(ab - h^2)}}{2} = \frac{a + b \pm \sqrt{(a - b)^2 + 4h^2}}{2} \qquad (6.2)$$

(The roots are distinct if $a \neq b$ or $h \neq 0$. The case $a = b$ and $h = 0$ needs no investigation, as it gives an equation of a circle.)

The equation $\lambda^2 - (a+b)\lambda + ab - h^2 = 0$ is called the *eigenvalue equation* of the matrix A.

6.2 Definitions and examples

DEFINITION 6.2.1 (Eigenvalue, eigenvector)
Let A be a complex square matrix. Then if λ is a complex number and X a *non–zero* complex column vector satisfying $AX = \lambda X$, we call X an *eigenvector* of A, while λ is called an *eigenvalue* of A. We also say that X is an eigenvector corresponding to the eigenvalue λ.

So in the above example P_1 and P_2 are eigenvectors corresponding to λ_1 and λ_2, respectively. We shall give an algorithm which starts from the eigenvalues of $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ and constructs a rotation matrix P such that P^tAP is diagonal.

As noted above, if λ is an eigenvalue of an $n \times n$ matrix A, with corresponding eigenvector X, then $(A - \lambda I_n)X = 0$, with $X \neq 0$, so $\det(A - \lambda I_n) = 0$ and there are at most n distinct eigenvalues of A.

Conversely if $\det(A - \lambda I_n) = 0$, then $(A - \lambda I_n)X = 0$ has a non–trivial solution X and so λ is an eigenvalue of A with X a corresponding eigenvector.

DEFINITION 6.2.2 (Characteristic equation, polynomial)
The equation $\det(A - \lambda I_n) = 0$ is called the *characteristic equation* of A, while the polynomial $\det(A - \lambda I_n)$ is called the *characteristic polynomial* of A. The characteristic polynomial of A is often denoted by $\mathrm{ch}_A(\lambda)$.

Hence the eigenvalues of A are the roots of the characteristic polynomial of A.

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, it is easily verified that the characteristic polynomial is $\lambda^2 - (\mathrm{trace}\, A)\lambda + \det A$, where $\mathrm{trace}\, A = a + d$ is the sum of the diagonal elements of A.

EXAMPLE 6.2.1 Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and find all eigenvectors.

Solution. The characteristic equation of A is $\lambda^2 - 4\lambda + 3 = 0$, or

$$(\lambda - 1)(\lambda - 3) = 0.$$

Hence $\lambda = 1$ or 3. The eigenvector equation $(A - \lambda I_n)X = 0$ reduces to

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or

$$\begin{aligned}(2-\lambda)x + y &= 0\\ x + (2-\lambda)y &= 0.\end{aligned}$$

Taking $\lambda = 1$ gives

$$\begin{aligned}x + y &= 0\\ x + y &= 0,\end{aligned}$$

which has solution $x = -y$, y arbitrary. Consequently the eigenvectors corresponding to $\lambda = 1$ are the vectors $\begin{bmatrix} -y \\ y \end{bmatrix}$, with $y \neq 0$.

Taking $\lambda = 3$ gives

$$\begin{aligned}-x + y &= 0\\ x - y &= 0,\end{aligned}$$

which has solution $x = y$, y arbitrary. Consequently the eigenvectors corresponding to $\lambda = 3$ are the vectors $\begin{bmatrix} y \\ y \end{bmatrix}$, with $y \neq 0$.

Our next result has wide applicability:

THEOREM 6.2.1 Let A be a 2×2 matrix having distinct eigenvalues λ_1 and λ_2 and corresponding eigenvectors X_1 and X_2. Let P be the matrix whose columns are X_1 and X_2, respectively. Then P is non–singular and

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Proof. Suppose $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$. We show that the system of homogeneous equations

$$xX_1 + yX_2 = 0$$

has only the trivial solution. Then by theorem 2.5.10 the matrix $P = [X_1|X_2]$ is non–singular. So assume

$$xX_1 + yX_2 = 0. \tag{6.3}$$

Then $A(xX_1 + yX_2) = A0 = 0$, so $x(AX_1) + y(AX_2) = 0$. Hence

$$x\lambda_1 X_1 + y\lambda_2 X_2 = 0. \tag{6.4}$$

Multiplying equation 6.3 by λ_1 and subtracting from equation 6.4 gives

$$(\lambda_2 - \lambda_1)yX_2 = 0.$$

Hence $y = 0$, as $(\lambda_2 - \lambda_1) \neq 0$ and $X_2 \neq 0$. Then from equation 6.3, $xX_1 = 0$ and hence $x = 0$.

Then the equations $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$ give

$$\begin{aligned} AP = A[X_1|X_2] = [AX_1|AX_2] &= [\lambda_1 X_1|\lambda_2 X_2] \\ &= [X_1|X_2]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \end{aligned}$$

so

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

EXAMPLE 6.2.2 Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ be the matrix of example 6.2.1. Then $X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors corresponding to eigenvalues 1 and 3, respectively. Hence if $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, we have

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

There are two immediate applications of theorem 6.2.1. The first is to the calculation of A^n: If $P^{-1}AP = \mathrm{diag}\,(\lambda_1, \lambda_2)$, then $A = P\mathrm{diag}\,(\lambda_1, \lambda_2)P^{-1}$ and

$$A^n = \left(P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}P^{-1}\right)^n = P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^n P^{-1} = P\begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}P^{-1}.$$

The second application is to solving a system of linear differential equations

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy, \end{aligned}$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix of real or complex numbers and x and y are functions of t. The system can be written in matrix form as $\dot{X} = AX$, where

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}.$$

We make the substitution $X = PY$, where $Y = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$. Then x_1 and y_1 are also functions of t and

$$\dot{X} = P\dot{Y} = AX = A(PY), \text{ so } \dot{Y} = (P^{-1}AP)Y = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} Y.$$

Hence $\dot{x_1} = \lambda_1 x_1$ and $\dot{y_1} = \lambda_2 y_1$.

These differential equations are well–known to have the solutions $x_1 = x_1(0)e^{\lambda_1 t}$ and $y_1 = y_1(0)e^{\lambda_2 t}$, where $x_1(0)$ is the value of x_1 when $t = 0$.

[If $\frac{dx}{dt} = kx$, where k is a constant, then

$$\frac{d}{dt}\left(e^{-kt}x\right) = -ke^{-kt}x + e^{-kt}\frac{dx}{dt} = -ke^{-kt}x + e^{-kt}kx = 0.$$

Hence $e^{-kt}x$ is constant, so $e^{-kt}x = e^{-k0}x(0) = x(0)$. Hence $x = x(0)e^{kt}$.]

However $\begin{bmatrix} x_1(0) \\ y_1(0) \end{bmatrix} = P^{-1}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$, so this determines $x_1(0)$ and $y_1(0)$ in terms of $x(0)$ and $y(0)$. Hence ultimately x and y are determined as explicit functions of t, using the equation $X = PY$.

EXAMPLE 6.2.3 Let $A = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}$. Use the eigenvalue method to derive an explicit formula for A^n and also solve the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 2x - 3y \\ \frac{dy}{dt} &= 4x - 5y, \end{aligned}$$

given $x = 7$ and $y = 13$ when $t = 0$.

Solution. The characteristic polynomial of A is $\lambda^2+3\lambda+2$ which has distinct roots $\lambda_1 = -1$ and $\lambda_2 = -2$. We find corresponding eigenvectors $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Hence if $P = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$, we have $P^{-1}AP = \text{diag}\,(-1, -2)$. Hence

$$\begin{aligned} A^n &= \left(P\text{diag}\,(-1, -2)P^{-1}\right)^n = P\text{diag}\,((-1)^n, (-2)^n)P^{-1} \\ &= \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & (-2)^n \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= (-1)^n \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \\
&= (-1)^n \begin{bmatrix} 1 & 3 \times 2^n \\ 1 & 4 \times 2^n \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \\
&= (-1)^n \begin{bmatrix} 4 - 3 \times 2^n & -3 + 3 \times 2^n \\ 4 - 4 \times 2^n & -3 + 4 \times 2^n \end{bmatrix}.
\end{aligned}
$$

To solve the differential equation system, make the substitution $X = PY$. Then $x = x_1 + 3y_1$, $y = x_1 + 4y_1$. The system then becomes

$$
\begin{aligned}
\dot{x}_1 &= -x_1 \\
\dot{y}_1 &= -2y_1,
\end{aligned}
$$

so $x_1 = x_1(0)e^{-t}$, $y_1 = y_1(0)e^{-2t}$. Now

$$
\begin{bmatrix} x_1(0) \\ y_1(0) \end{bmatrix} = P^{-1} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 13 \end{bmatrix} = \begin{bmatrix} -11 \\ 6 \end{bmatrix},
$$

so $x_1 = -11e^{-t}$ and $y_1 = 6e^{-2t}$. Hence $x = -11e^{-t} + 3(6e^{-2t}) = -11e^{-t} + 18e^{-2t}$, $y = -11e^{-t} + 4(6e^{-2t}) = -11e^{-t} + 24e^{-2t}$.

For a more complicated example we solve a system of *inhomogeneous* recurrence relations.

EXAMPLE 6.2.4 Solve the system of recurrence relations

$$
\begin{aligned}
x_{n+1} &= 2x_n - y_n - 1 \\
y_{n+1} &= -x_n + 2y_n + 2,
\end{aligned}
$$

given that $x_0 = 0$ and $y_0 = -1$.

Solution. The system can be written in matrix form as

$$X_{n+1} = AX_n + B,$$

where

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

It is then an easy induction to prove that

$$X_n = A^n X_0 + (A^{n-1} + \cdots + A + I_2)B. \tag{6.5}$$

Also it is easy to verify by the eigenvalue method that

$$A^n = \frac{1}{2}\begin{bmatrix} 1+3^n & 1-3^n \\ 1-3^n & 1+3^n \end{bmatrix} = \frac{1}{2}U + \frac{3^n}{2}V,$$

where $U = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $V = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Hence

$$\begin{aligned} A^{n-1} + \cdots + A + I_2 &= \frac{n}{2}U + \frac{(3^{n-1} + \cdots + 3 + 1)}{2}V \\ &= \frac{n}{2}U + \frac{(3^{n-1} - 1)}{4}V. \end{aligned}$$

Then equation 6.5 gives

$$X_n = \left(\frac{1}{2}U + \frac{3^n}{2}V\right)\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \left(\frac{n}{2}U + \frac{(3^{n-1}-1)}{4}V\right)\begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

which simplifies to

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} (2n+1-3^n)/4 \\ (2n-5+3^n)/4 \end{bmatrix}.$$

Hence $x_n = (2n-1+3^n)/4$ and $y_n = (2n-5+3^n)/4$.

REMARK 6.2.1 If $(A - I_2)^{-1}$ existed (that is, if $\det(A - I_2) \neq 0$, or equivalently, if 1 is not an eigenvalue of A), then we could have used the formula

$$A^{n-1} + \cdots + A + I_2 = (A^n - I_2)(A - I_2)^{-1}. \qquad (6.6)$$

However the eigenvalues of A are 1 and 3 in the above problem, so formula 6.6 cannot be used there.

Our discussion of eigenvalues and eigenvectors has been limited to 2×2 matrices. The discussion is more complicated for matrices of size greater than two and is best left to a second course in linear algebra. Nevertheless the following result is a useful generalization of theorem 6.2.1. The reader is referred to [28, page 350] for a proof.

THEOREM 6.2.2 Let A be an $n \times n$ matrix having distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $X_1, \ldots, X_n$. Let P be the matrix whose columns are respectively $X_1, \ldots, X_n$. Then P is non–singular and

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Another useful result which covers the case where there are multiple eigenvalues is the following (The reader is referred to [28, pages 351–352] for a proof):

THEOREM 6.2.3 Suppose the characteristic polynomial of A has the factorization

$$\det(\lambda I_n - A) = (\lambda - c_1)^{n_1} \cdots (\lambda - c_t)^{n_t},$$

where $c_1, \ldots, c_t$ are the distinct eigenvalues of A. Suppose that for $i = 1, \ldots, t$, we have nullity $(c_i I_n - A) = n_i$. For each i, choose a basis $X_{i1}, \ldots, X_{in_i}$ for the *eigenspace* $N(c_i I_n - A)$. Then the matrix

$$P = [X_{11}|\cdots|X_{1n_1}|\cdots|X_{t1}|\cdots|X_{tn_t}]$$

is non–singular and $P^{-1}AP$ is the following diagonal matrix

$$P^{-1}AP = \begin{bmatrix} c_1 I_{n_1} & 0 & \cdots & 0 \\ 0 & c_2 I_{n_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c_t I_{n_t} \end{bmatrix}.$$

(The notation means that on the diagonal there are n_1 elements c_1, followed by n_2 elements $c_2, \ldots$, n_t elements c_t.)

6.3 PROBLEMS

1. Let $A = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$. Find a non–singular matrix P such that $P^{-1}AP = \operatorname{diag}(1,\, 3)$ and hence prove that

$$A^n = \frac{3^n - 1}{2}A + \frac{3 - 3^n}{2}I_2.$$

2. If $A = \begin{bmatrix} 0.6 & 0.8 \\ 0.4 & 0.2 \end{bmatrix}$, prove that A^n tends to a limiting matrix

$$\begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}$$

as $n \to \infty$.

3. Solve the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 3x - 2y \\ \frac{dy}{dt} &= 5x - 4y, \end{aligned}$$

given $x = 13$ and $y = 22$ when $t = 0$.
[Answer: $x = 7e^t + 6e^{-2t}$, $y = 7e^t + 15e^{-2t}$.]

4. Solve the system of recurrence relations

$$\begin{aligned} x_{n+1} &= 3x_n - y_n \\ y_{n+1} &= -x_n + 3y_n, \end{aligned}$$

given that $x_0 = 1$ and $y_0 = 2$.
[Answer: $x_n = 2^{n-1}(3 - 2^n)$, $y_n = 2^{n-1}(3 + 2^n)$.]

5. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a real or complex matrix with distinct eigenvalues λ_1, λ_2 and corresponding eigenvectors X_1, X_2. Also let $P = [X_1|X_2]$.
(a) Prove that the system of recurrence relations

$$\begin{aligned} x_{n+1} &= ax_n + by_n \\ y_{n+1} &= cx_n + dy_n \end{aligned}$$

has the solution

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \alpha\lambda_1^n X_1 + \beta\lambda_2^n X_2,$$

where α and β are determined by the equation

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = P^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

(b) Prove that the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}$$

has the solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha e^{\lambda_1 t} X_1 + \beta e^{\lambda_2 t} X_2,$$

where α and β are determined by the equation

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = P^{-1} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}.$$

6. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a real matrix with non–real eigenvalues $\lambda = a + ib$ and $\overline{\lambda} = a - ib$, with corresponding eigenvectors $X = U + iV$ and $\overline{X} = U - iV$, where U and V are real vectors. Also let P be the real matrix defined by $P = [U|V]$. Finally let $a + ib = re^{i\theta}$, where $r > 0$ and θ is real.

   (a) Prove that

$$\begin{aligned} AU &= aU - bV \\ AV &= bU + aV. \end{aligned}$$

   (b) Deduce that

$$P^{-1}AP = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

   (c) Prove that the system of recurrence relations

$$\begin{aligned} x_{n+1} &= a_{11}x_n + a_{12}y_n \\ y_{n+1} &= a_{21}x_n + a_{22}y_n \end{aligned}$$

   has the solution

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = r^n\{(\alpha U + \beta V)\cos n\theta + (\beta U - \alpha V)\sin n\theta\},$$

   where α and β are determined by the equation

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = P^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

   (d) Prove that the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}$$

has the solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{at}\{(\alpha U + \beta V)\cos bt + (\beta U - \alpha V)\sin bt\},$$

where α and β are determined by the equation

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = P^{-1}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix}.$$

[Hint: Let $\begin{bmatrix} x \\ y \end{bmatrix} = P\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$. Also let $z = x_1 + iy_1$. Prove that

$$\dot{z} = (a - ib)z$$

and deduce that

$$x_1 + iy_1 = e^{at}(\alpha + i\beta)(\cos bt + i\sin bt).$$

Then equate real and imaginary parts to solve for x_1, y_1 and hence x, y.]

7. (The case of repeated eigenvalues.) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and suppose that the characteristic polynomial of A, $\lambda^2 - (a+d)\lambda + (ad - bc)$, has a repeated root α. Also assume that $A \neq \alpha I_2$. Let $B = A - \alpha I_2$.

   (i) Prove that $(a - d)^2 + 4bc = 0$.

   (ii) Prove that $B^2 = 0$.

   (iii) Prove that $BX_2 \neq 0$ for some vector X_2; indeed, show that X_2 can be taken to be $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   (iv) Let $X_1 = BX_2$. Prove that $P = [X_1|X_2]$ is non–singular,

   $$AX_1 = \alpha X_1 \text{ and } AX_2 = \alpha X_2 + X_1$$

   and deduce that

   $$P^{-1}AP = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}.$$

8. Use the previous result to solve system of the differential equations

$$\begin{aligned} \frac{dx}{dt} &= 4x - y \\ \frac{dy}{dt} &= 4x + 8y, \end{aligned}$$

given that $x = 1 = y$ when $t = 0$.

[To solve the differential equation

$$\frac{dx}{dt} - kx = f(t), \quad k \text{ a constant},$$

multiply throughout by e^{-kt}, thereby converting the left–hand side to $\frac{dx}{dt}(e^{-kt}x)$.]

[Answer: $x = (1 - 3t)e^{6t}$, $y = (1 + 6t)e^{6t}$.]

9. Let

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}.$$

(a) Verify that $\det(\lambda I_3 - A)$, the characteristic polynomial of A, is given by

$$(\lambda - 1)\lambda(\lambda - \frac{1}{4}).$$

(b) Find a non–singular matrix P such that $P^{-1}AP = \text{ diag}\,(1,\ 0,\ \frac{1}{4})$.

(c) Prove that

$$A^n = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{1}{3 \cdot 4^n}\begin{bmatrix} 2 & 2 & -4 \\ -1 & -1 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

if $n \geq 1$.

10. Let

$$A = \begin{bmatrix} 5 & 2 & -2 \\ 2 & 5 & -2 \\ -2 & -2 & 5 \end{bmatrix}.$$

(a) Verify that $\det(\lambda I_3 - A)$, the characteristic polynomial of A, is given by

$$(\lambda - 3)^2(\lambda - 9).$$

(b) Find a non–singular matrix P such that $P^{-1}AP = \text{ diag}\,(3,\ 3,\ 9)$.