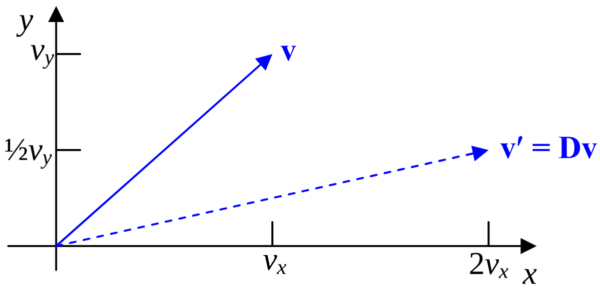
10. Eigenvectors and Eigenvalues

10.0 *Motivation:*

Consider a deformation matrix, $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & ½ \end{pmatrix}$, applied to a vector **v**



$v'_x = 2v_x$
$v'_y = ½v_x$

Under the deformation **D** a general vector **v** is rotated and stretched.

Two special vectors, **i** and **j**, are not rotated. They are only stretched:

$$\hat{\mathbf{i}} \to 2\hat{\mathbf{i}} \qquad \hat{\mathbf{j}} \to ½\hat{\mathbf{j}}$$

Now consider the deformation, $\mathbf{F} = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$.

Does it leave any vector unrotated?

Try solving the equation $\mathbf{Fv} = \lambda\mathbf{v}$, with λ a scalar constant, i.e.

$$\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \lambda\begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\equiv \begin{cases} \frac{5}{4}v_x + \frac{3}{4}v_y = \lambda v_x \\ \frac{3}{4}v_x + \frac{5}{4}v_y = \lambda v_y \end{cases}$$

Test for a solution with $v_x = 1$: $\quad 5 + 3v_y = 4\lambda$
$3 + 5v_y = 4\lambda\, v_y$

So $\quad 3 + 5v_y = (5 + 3v_y)\, v_y$
$\therefore\ 3 = 3v_y^2$ and so $v_y = \pm 1$

When $v_y = +1$, $\qquad \lambda = 2$

When $v_y = -1$, $\qquad \lambda = ½$

So, $\quad \mathbf{v} = (1, 1) \to \mathbf{v}' = 2(1, 1) = 2\mathbf{v}$
$\mathbf{v} = (1, -1) \to \mathbf{v}' = ½(1, -1) = ½\mathbf{v}$

<u>**Conclusion:**</u> **F** and **D** are the <u>**same**</u> deformation apart from orientation with respect to the axes.

		F	**D**
Matrix		$\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & ½ \end{pmatrix}$
Trace		2½	2½
Determinant		1	1
Eigenvalues	λ_1	2	2
	λ_2	½	½
Eigenvectors	$\mathbf{v}_1$	$\frac{1}{\sqrt{2}}(1, 1)$	$(1, 0)$
(normalised)	$\mathbf{v}_2$	$\frac{1}{\sqrt{2}}(1, -1)$	$(0, 1)$

10.1 *Motivation in Physics: In Quantum Mechanics,*

- **A physical system is described by a state vector**
- **A measurement is a matrix operator**
- **The outcome of a measurement (a physical observable) is an eigenvalue of the matrix**
- **The system is left after the measurement in an eigenstate, i.e. with a state vector which is an eigenvector of the matrix.**

10.2 *Definitions*

- **In this section, a vector will be a column vector, an *n* × 1 matrix**

E.g. $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$

- **If A is an *n* × *n* matrix, then Ax is also a column vector**

E.g. $\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

This is $\mathbf{y} = \mathbf{Ax}$, or $y_i = \sum_{j=1}^{n} a_{ij} x_j$, or $y_i = a_{ij}x_j$

<u>This is *n* simultaneous equations.</u>

- **The Eigenvalue Equation:**

  Let $\mathbf{Ax} = \lambda \mathbf{x}$ for some constant λ

  Then **x** is an **eigenvector** of **A** with **eigenvalue** λ.

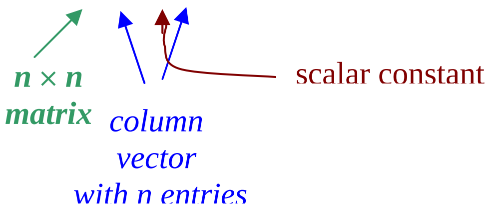
- **Solving $\mathbf{Ax} = \lambda \mathbf{x}$ is called "solving the eigenvalue problem".**

- **A normalised eigenvector is the unit vector in that direction, $\hat{\mathbf{x}} = \dfrac{\mathbf{x}}{|\mathbf{x}|}$**

10.3 *Eigen = Proper, Own, Particular . . .*

Suppose that **Ax** is proportional to **x**

$$\mathbf{Ax} = \lambda \mathbf{x} \qquad \text{for some constant } \lambda$$



Then **x** is an **eigenvector** of **A**, with **eigenvalue** of λ

- Eigenvectors defined in direction, not in length, for if **x** obeys $\mathbf{Ax} = \lambda \mathbf{x}$, then so does any $c\mathbf{x}$ where c is a scalar.

- Normalised eigenvectors are of unit length, i.e. $\hat{\mathbf{x}} = \dfrac{\mathbf{x}}{|\mathbf{x}|}$

Example:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = +1 \times \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1 \times \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

10.4 *Solving for Eigenvalues and Eigenvectors*

1. The Clumsy Way: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$

This is two simultaneous equations,

$$\begin{cases} 0x + 1y = \lambda x \\ 1x + 0y = \lambda y \end{cases}$$

From the second, $x = \lambda y$, **so from the first,** $y = \lambda^2 y$

$\Rightarrow \lambda^2 = 1,$

$\therefore\ \lambda = \pm 1$

So eigenvectors are

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *or* $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **with eigenvalue** $+1$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *or* $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **with eigenvalue** -1

2. The Neat Way: Rewrite $\lambda\mathbf{x}$ as

$$\lambda\mathbf{x} = \lambda\mathbf{E}\mathbf{x} = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Then $\mathbf{Ax} = \lambda\mathbf{x}$ becomes $(\mathbf{A} - \lambda\mathbf{E})\mathbf{x} = 0$, i.e.

$$\begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & & 0 \\ \vdots & & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} - \lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

"Size" = 0

i.e. $det(\mathbf{A} - \lambda\mathbf{E}) = 0$

Example from above becomes: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$

Taking determinant, $\lambda^2 - 1 = 0$, **so** $\lambda = \pm 1$ **as before.**

Now to find the eigenvectors:

For each eigenvalue λ, solve

$A\mathbf{x} = \lambda\mathbf{x}$ for **x**

Thus, for $\lambda = +1$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = +1\begin{pmatrix} x \\ y \end{pmatrix}$$

i.e.

$$\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

So, $y = x$, and $x = y$, so $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

And similarly, for $\lambda = -1$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$$

i.e.

$$\begin{pmatrix} y \\ x \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}$$

So, $y = -x$, and $x = -y$, so $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

10.5 *Normalising Eigenvectors*

- Real vectors: $\mathbf{a} = (a_x, a_y, a_z)$ has length^2 = $|\mathbf{a}|^2 = a_x^2 + a_y^2 + a_z^2$

$$= \begin{pmatrix} a_x & a_y & a_z \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

$$= \mathbf{a}^{\mathrm{T}}\mathbf{a}$$

So the normalised real eigenvector, when **a** is an eigenvector, is

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{\sqrt{\mathbf{a}^{\mathrm{T}}\mathbf{a}}}$$

- Complex vectors: $\mathbf{c} = (c_x, c_y, c_z)$ has length^2 = $|\mathbf{c}|^2 = c_x^* c_x + c_y^* c_y + c_z^* c_z$

$$= \begin{pmatrix} c_x^* & c_y^* & c_z^* \end{pmatrix} \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix}$$

$$= \mathbf{c}^{\dagger}\mathbf{c}$$

So the normalised complex eigenvector, when **c** is an eigenvector, is

$$\hat{\mathbf{c}} = \frac{\mathbf{c}}{\sqrt{\mathbf{c}^{\dagger}\mathbf{c}}}$$

10.5 ***Diagonalising Matrices***

Consider $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$

i.e.
$$\mathbf{A}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$
$$\mathbf{A}\begin{pmatrix} x_2 \\ y2 \end{pmatrix} = \lambda_2 \begin{pmatrix} x_2 \\ y2 \end{pmatrix}$$

We can tidy this up as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 \\ \lambda_1 y_1 & \lambda_2 y_2 \end{pmatrix}$$

That is,

$$\mathbf{AX} = \mathbf{X\Lambda}$$

where **X** is assembled out of the eigenvectors
and **Λ** out of the eigenvalues of **A**

Hence, multiplying from the left by $\mathbf{X}^{-1}$,

$$\mathbf{X}^{-1}\mathbf{AX} = \Lambda$$

i.e., **X diagonalises A**

10.6 ***Summary of Properties***

- **X** is matrix of eigenvectors of **A**
- **Λ** is diagonal matrix of eigenvalues of **A**
- This implies that an $n \times n$ matrix **A** has n eigenvalues $\lambda_1, \ldots \lambda_n$ (needn't all be different).
- $\det \mathbf{A} = \det \mathbf{\Lambda} = \lambda_1\lambda_2\lambda_3 \ldots \lambda_n = \prod_{i=1}^{n} \lambda_i$
- $\text{tr}\, \mathbf{A} = \text{tr} = \lambda_1 + \lambda_3 + \ldots + \lambda_n = \sum_{i=1}^{n} \lambda_i$
- $$\begin{aligned} \mathbf{E} &= \mathbf{X}^{-1}\mathbf{X} \\ &= \mathbf{X}^{-1}\mathbf{A}\,\mathbf{A}^{-1}\mathbf{X} \\ &= \mathbf{X}^{-1}\mathbf{A}\,\mathbf{X}\,\mathbf{X}^{-1}\mathbf{A}^{-1}\mathbf{X} \\ &= \quad \mathbf{\Lambda} \quad\quad \mathbf{\Lambda}^{-1} \quad = \mathbf{E} \end{aligned}$$

Example: Given $\mathbf{X}^{-1}\mathbf{A}\,\mathbf{X} = \mathbf{\Lambda}$ where $\mathbf{\Lambda}$ is diagonal,

$\mathbf{A} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, find **X** and **Λ:**

Eigenvectors: Solve $det(\mathbf{A} - (\lambda\delta_{ij})) = 0$,

i.e. $det\begin{pmatrix} 0-\lambda & -i \\ i & 0-\lambda \end{pmatrix} = 0 = \lambda^2 - (-i \times i)$

so that $\lambda^2 = 1$, $\lambda = \pm 1$

Eigenvectors: For each eigenvalue, try (x, y):

For $\lambda_1 = 1$, $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} -iy \\ ix \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

so $-iy = x$, satisfied by $\mathbf{x}^{(2)} = (1, i)$

For $\lambda_2 = -1$, $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} -iy \\ ix \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$

so $-iy = -x$, satisfied by $\mathbf{x}^{(2)} = (1, -i)$

Normalising, $\left|\mathbf{x}^{(1)}\right|^2 = \mathbf{x}^{(1)\dagger}\mathbf{x}^{(1)} = 2$

$\left|\mathbf{x}^{(2)}\right|^2 = \mathbf{x}^{(2)\dagger}\mathbf{x}^{(2)} = 2$

So $\mathbf{X} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ **and** $\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

10.7 *Some Special Cases*

- $\mathbf{A}$ Hermitian (i.e. $\mathbf{A} = \mathbf{A}^{\dagger} = \mathbf{A}^{T*}$)
  - **Eigenvalues Real**
  - **Eigenvectors Perpendicular**

    $\Rightarrow$ $\exists\, n$ independent eigenvectors which are orthonormal

    $\Rightarrow$ $\mathbf{X}^{-1}\,\mathbf{A}\,\mathbf{X} = \Lambda$ **with X unitary**

- $\mathbf{A}$ Real Symmetric (i.e. $\mathbf{A} = \mathbf{A}^{*} = \mathbf{A}^{T}$)

  Special case of Hermitian

- $\mathbf{A}$ Diagonal

  $\mathbf{A}\,\mathbf{x}^{(n)} = \lambda_n\,\mathbf{x}^{(n)}$ becomes

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & \ddots & & & \vdots \\ \vdots & & a_{jj} & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ a_{jj} \\ \vdots \\ 0 \end{pmatrix} = \lambda_j \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

so that a_{jj} is the j^{th} eigenvalue

and $\hat{j}$ is the j^{th} eigenvector