5. Vector Fields

5.0 *Vector fields are related to scalar fields.*

The *Position Vector*, the vector from the origin to the point (x, y), is $\mathbf{r} = (x, y)$

The *Scalar Field* is a scalar function of position, $\Phi(x, y) = \Phi(\mathbf{r})$

e.g. $T(\mathbf{r})$ – temperature as a function of position

e.g. $V(\mathbf{r})$ – electrostatic potential as a function of position

The *Vector Field* is a vector function of position, $\mathbf{A}(x, y) = \mathbf{A}(\mathbf{r})$

It has three component scalar fields, $\mathbf{A}(\mathbf{r}) = (A_x(\mathbf{r}), A_y(\mathbf{r}), A_z(\mathbf{r}))$

e.g. $\mathbf{v}(\mathbf{r})$ – velocity as a function of position

e.g. $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ – electric and magnetic fields.

5.1 *Div, Grad and Curl*

Divergence, Gradient and Curl are **properties** of scalar and vector fields.

- **Gradient:** The lines of steepest slope of a scalar field, $\Phi(x, y, z)$, are a vector field, grad $\Phi\ (x, y, z)$

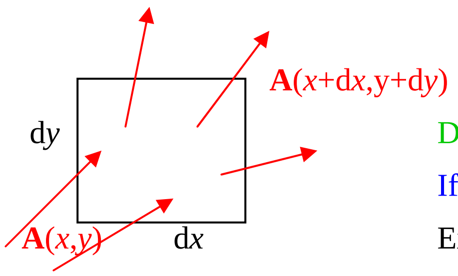
  e.g. a mountain has an altitude, h, at each point, $h(x, y, z)$.
  Contour lines are curves along which $h(x, y, z)$ = constant
  grad h is everywhere $\perp$ contours lines and is larger when contour lines are closer spaced.
  So it has magnitude and direction (vector) as a function of position (vector field)

- **Divergence:** The gradient vector field can converge onto a mountain peak. It can diverge from a hollow.

Positive divergence   Negative divergence

There is divergence at every point.

Consider an element dx × dy (× dz) at (x, y).



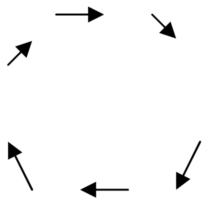
Does more come out than goes in?

If so, then divergence is +ve at (x, y)

Excess is div**A**dxdy

div **A**(**r**) has a scalar value at each point, so it is a scalar field, and it is a scalar function of a vector field.

- **Curl or Rotation:** Does a vector field go round and round?
  I.e., can there be a vortex?



grad $\Phi(\mathbf{r})$ does not go round and round.

(You cannot walk uphill only, yet return to your starting-point.)

So curl grad $\Phi(\mathbf{r}) = 0$

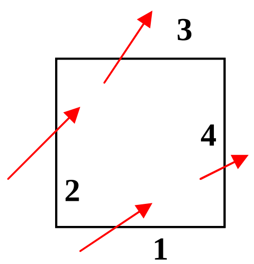
A velocity field can go round and round.
(E.g. the velocity field of a turntable or a typhoon.)

There is curl at every point.
Curl has a value at every point
Curl has an axis (i.e. a direction)

So curl $\mathbf{A}(\mathbf{r})$ is a vector field, and it is a vector function of a vector field

5.2 *Mathematical expressions for Div, Grad and Curl*

- **Mathematic Divergence:**



Consider a vector field $\mathbf{A}(x, y)$

at the point (x, y)

What goes into the elemental square $dx \times dy$?

Side 1: **IN:** $\mathbf{A}_y(x, y)\, dx$

Side 2: **IN:** $\mathbf{A}_x(x, y)\, dy$

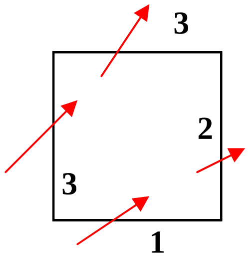
Side 3: **OUT:** $\mathbf{A}_y(x, y + dy)\, dx = \left(\mathbf{A}_y(x,y) + \frac{\partial \mathbf{A}_y}{\partial y} dy\right) dx$

Side 4: **OUT:** $\mathbf{A}_x(x + dx, y)\, dy = \left(\mathbf{A}_x(x,y) + \frac{\partial \mathbf{A}_x}{\partial x} dx\right) dy$

Total: **We want 3 + 4 – 1 – 2 =** $\left(\frac{\partial \mathbf{A}_x}{\partial x} + \frac{\partial \mathbf{A}_y}{\partial y}\right) dxdy$

Per unit area, $\quad \mathrm{div}\mathbf{A} = \left(\frac{\partial \mathbf{A}_x}{\partial x} + \frac{\partial \mathbf{A}_y}{\partial y}\right)$

- **Mathematic Curl:**



Consider a vector field $\mathbf{A}(x, y)$

at the point (x, y)

Walk round the elemental square $dx \times dy$.

Along Side 1: $+\ \mathbf{A}_x(x, y)\, dx$

Along Side 2: $+\ \mathbf{A}_y(x + dx, y)\, dy = \left(\mathbf{A}_y(x, y) + \frac{\partial \mathbf{A}_y}{\partial x} dx\right) dy$

Along Side 3: $-\ \mathbf{A}_x(x, y + dy)\, dx = \left(\mathbf{A}_x(x, y) + \frac{\partial \mathbf{A}_x}{\partial y} dy\right) dx$

Along Side 4: $-\ \mathbf{A}_y(x, y)\, dy =$

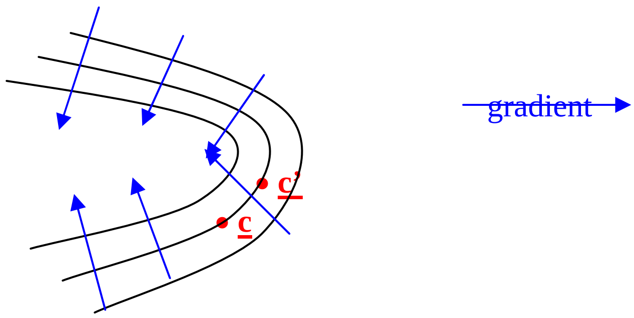
Total: **We want 3 + 4 – 1 – 2 =** $\left(\frac{\partial \mathbf{A}_y}{\partial x} + \frac{\partial \mathbf{A}_x}{\partial y}\right) dxdy$

This is about the z-axis, so it is the z component of curl.

Repeat on the $dydz$, $dzdx$ loops, or rotate subscripts, for x and y components.

Per unit area, $\text{curl}\mathbf{A} = \left(\cdots, \cdots, \frac{\partial \mathbf{A}_y}{\partial x} - \frac{\partial \mathbf{A}_x}{\partial y}\right)$

- **Mathematic Gradient:**



contour lines, or in 3D, *contour surfaces*

equivalent to equipotential surfaces, isotherms, etc

Let **c** and **c'** be two points on the surface $\Phi(\mathbf{r}) = c$, separated by $d\mathbf{r}$

Let **c** be at (x, y, z), then **c'** is at $(x + dx, y + dy, z + dz)$,

Then $\Phi(x, y, z) = c$ and $\Phi(x + dx, y + dy, z + dz) = c$

So $d\Phi = \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial z}dz = 0$

But this is $\left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}\right) \cdot (dx, dy, dz)$

So grad $\Phi(x, y, z) = \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}\right)$

5.3 *The* Del *or* Nabla *Operator* ∇

- **Collect the mathematical results:**

$$\text{grad}\Phi = \left(\frac{\partial \mathbf{A}_x}{\partial x}, \frac{\partial \mathbf{A}_y}{\partial y}, \frac{\partial \mathbf{A}_z}{\partial z} \right)$$

$$\text{div}\mathbf{A} = \frac{\partial \mathbf{A}_x}{\partial x} + \frac{\partial \mathbf{A}_y}{\partial y} + \frac{\partial \mathbf{A}_z}{\partial z}$$

$$\text{curl}\mathbf{A} = \left(\frac{\partial \mathbf{A}_z}{\partial y} - \frac{\partial \mathbf{A}_y}{\partial z}, \frac{\partial \mathbf{A}_x}{\partial z} - \frac{\partial \mathbf{A}_z}{\partial x}, \frac{\partial \mathbf{A}_y}{\partial x} - \frac{\partial \mathbf{A}_x}{\partial y} \right)$$

We note the common factor,

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

∇ is a vector operator, with

grad $\Phi = \nabla\ \Phi$

div $\mathbf{A} = \nabla\ .\ \mathbf{A}$

curl $\mathbf{A} = \nabla \times \Phi$

Recall that $\nabla \times \mathbf{A}$ *is*

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

- **Nine Combinations of ∇:**

grad grad Φ	**NO**
grad div $\mathbf{A} = \nabla\ \nabla.\ \mathbf{A}$	**Exists**
grad curl $\mathbf{A}$	**NO**
div grad $\Phi = \nabla.\nabla\ \Phi = \nabla^2\ \Phi$	**Important**
div div $\mathbf{A}$	**NO**
div curl $\mathbf{A} = \nabla.\nabla \times \mathbf{A} = 0$	**Identity**
curl grad $\Phi = \nabla \times \nabla\ \Phi = 0$	**Identity**
curl div $\mathbf{A}$	**NO**
curl curl $\mathbf{A} = \nabla \times \nabla \times \mathbf{A} = \nabla\ \nabla.\ \mathbf{A} - \nabla^2\ \mathbf{A}$	**Exists**

- <u>**Some Properties of ∇:**</u>

$\nabla \times (\Phi \mathbf{A}) = (\nabla\Phi) \times \mathbf{A} = \Phi\,(\nabla \times \mathbf{A})$

$\nabla . (\mathbf{A} \times \mathbf{B}) = \mathbf{B} . (\nabla \times \mathbf{A}) - \mathbf{A} . (\nabla \times \mathbf{B})$

$\nabla . \mathbf{r} = 3$ note that $\mathbf{r} = (x, y, z)$

$\nabla . (r^n\, \mathbf{r}) = (n + 3)\, r^n$ note that r is the modulus of $\mathbf{r}$

$\nabla . (\Phi \mathbf{A}) = \nabla\Phi . \mathbf{A} + \Phi\nabla . \mathbf{A}$