3. Multiple Integration

3.0 ***Multiple integrals are common things in physics, e.g., area or volume. They can be done in Cartesian co-ordinates but they are often much simpler in polar co-ordinates.***

3.1 *Double Integrals*

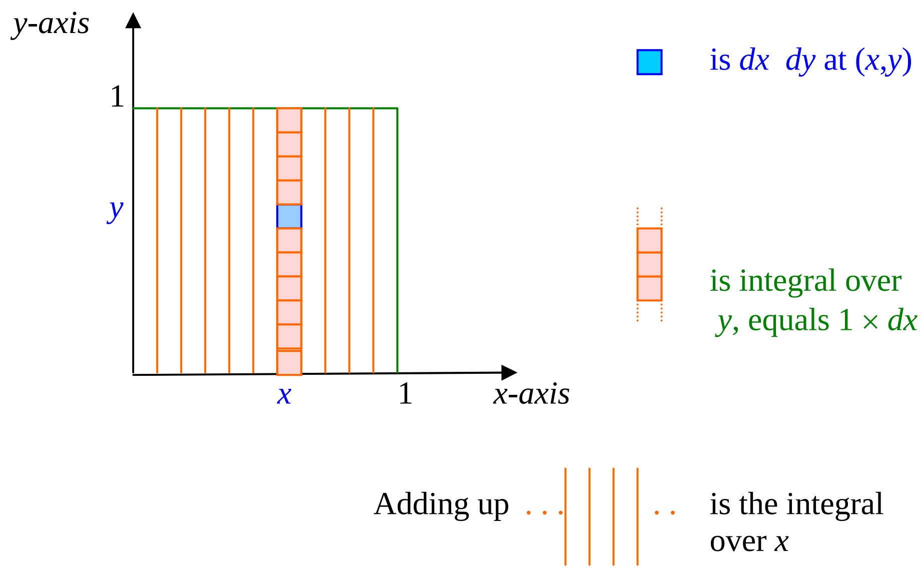
Cartesian: $$I = \int_{y=y_1}^{y_2}\int_{x=x_1}^{x_2} f(x,y)\,dx\,dy$$

often written: $$I = \int_{y=y_1}^{y_2} dy \int_{x=x_1}^{x_2} dx\; f(x,y)$$

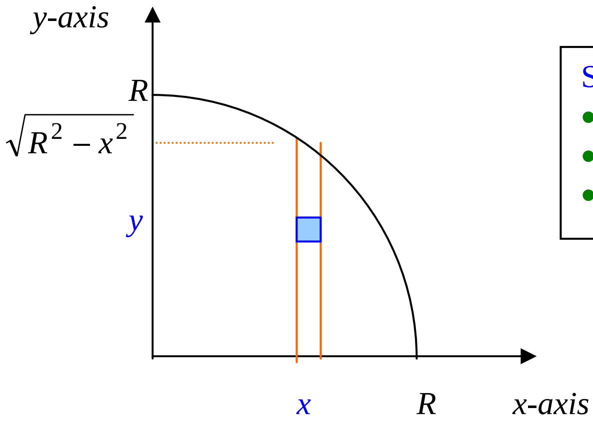
- It is conventional to do the inner integral first
- The limits, x_1, x_2, y_1, y_2, need not be constants

Examples

1. $$\int_0^1\int_0^1 dydx = y\Big]_0^1 \times \int_0^1 dx = y\Big]_0^1 \times x\Big]_0^1 = 1\times 1 = 1$$



2. $\int_0^R dx \int_0^{\sqrt{R^2-x^2}} dy = \int_0^R dx\, y\Big]_0^{\sqrt{R^2-x^2}} = \int_0^R dx \sqrt{R^2-x^2} = \frac{1}{4}\pi R^2$

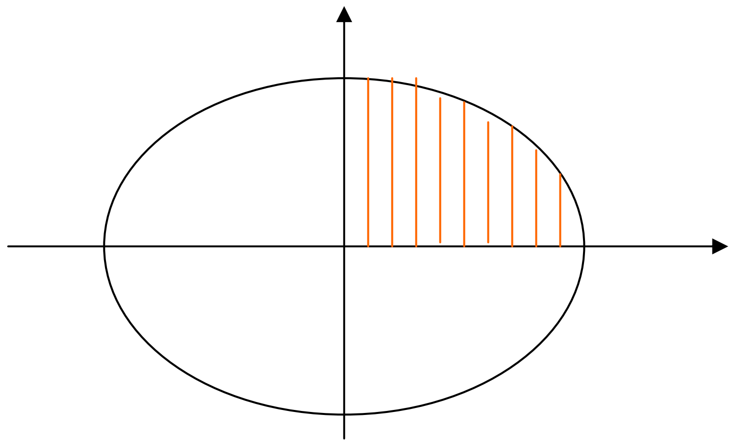


Same as example 1:

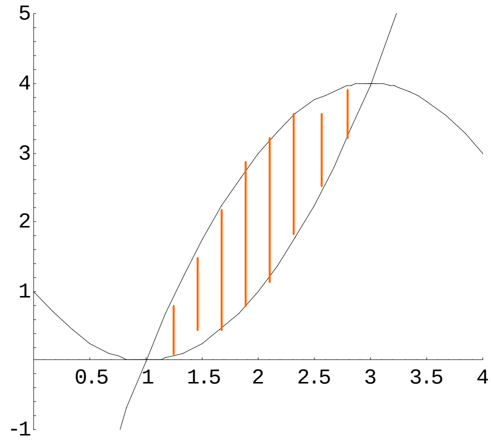
- Pick an x
- Integrate over y
- Integrate over x

3. **An ellipse** is $\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$

So area of one quadrant is $\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} dydx = \quad \cdots \quad = \frac{1}{4}\pi ab$



4. $$\int_1^3 dx \int_{(x-1)^2}^{4-(x-3)^2} dy = -2\int_1^3 dx(x^2 - 4x + 3) = \frac{8}{3}$$



This is the area of the plane figure bounded by the curves,
$y = (x - 1)^2$ and $y = 4 - (x - 3)^2$, which intersect at (1, 0) and (3, 4).

Those examples all gave Areas.
Now, for Volumes:

5. $$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2 + y^2}\; e^{-\lambda\sqrt{x^2+y^2}}\, dxdy$$ λ +ve constant

$x \to r\cos\theta$
$y \to r\sin\theta$

$$I = \int_0^{2\pi}\int_0^{\infty} r\, e^{-\lambda r}\; rdrd\theta$$

$$= 2\pi r^2 \frac{e^{-\lambda r}}{-\lambda}\Bigg]_{r=0}^{\infty} - \int_0^{\infty} 4\pi r \frac{e^{-\lambda r}}{-\lambda}\, drd\theta$$

'By Parts':
$u = r^2 \quad v' = e^{-\lambda r}$
$\int uv' = uv - \int u'v$

$$= 0 + \frac{4\pi}{\lambda}\int_0^{\infty} r\, e^{-\lambda r}\; drd\theta$$

'By Parts':
$u = r \quad v' = e^{-\lambda r}$
$\int uv' = uv - \int u'v$

$$= \frac{4\pi}{\lambda} r \frac{e^{-\lambda r}}{-\lambda}\Bigg]_{r=0}^{\infty} - \frac{4\pi}{\lambda}\int_0^{\infty} \frac{e^{-\lambda r}}{-\lambda}\, drd\theta$$

$$= 0 + \frac{4\pi}{\lambda^2}\int_0^{\infty} e^{-\lambda r}\; dr = \frac{4\pi}{\lambda^2}\frac{e^{-\lambda r}}{-\lambda}\Bigg]_0^{\infty} = \frac{4\pi}{\lambda^3}$$

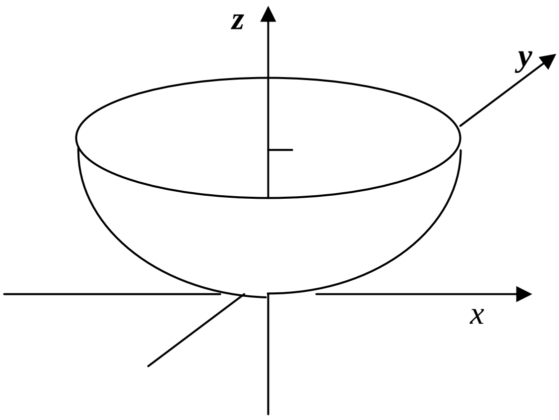
3.2 *Triple Integrals*

Cartesian: $$I = \int_{z=z_1}^{z_2}\int_{y=y_1}^{y_2}\int_{x=x_1}^{x_2} f(x,y,z)\,dxdydz$$

Triple Integral Examples

6. Volume of a Parabolic Dish

$z = x^2 + y^2$



Let $R = \sqrt{z}$

Then $$V = 4\int_{z=0}^{1}\int_{y=0}^{R}\int_{x=0}^{\sqrt{R^2-y^2}} dxdydz$$

Go to cylindrical polar co-ordinates: $x \to \rho\cos\theta,\ y \to \rho\sin\theta,\ z \to z,\ dxdydz \to \rho$

so that $$V = 4\int_{z=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\rho=0}^{R} \rho\, d\rho\, d\theta\, dz = \frac{\pi}{2}$$

7. Volume of a Sphere

$$x^2 + y^2 + z^2 = R^2$$

$$V = 8\int_{z=0}^{R}\int_{y=0}^{\sqrt{R^2-z^2}}\int_{x=0}^{\sqrt{R^2-y^2-z^2}} dxdydz$$

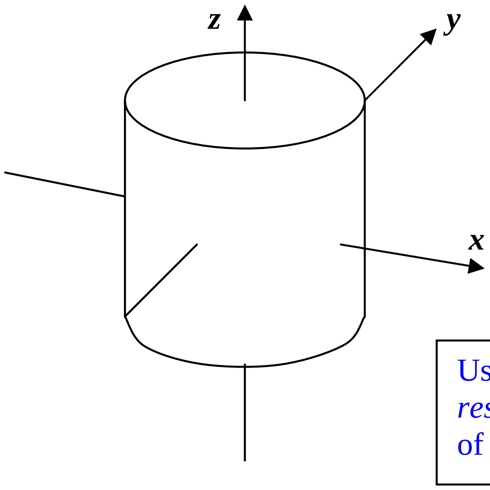
Go to spherical polar co-ordinates:
$x \to r\sin\theta\cos\varphi,\ \ y \to r\sin\theta\sin\varphi,\ \ z \to r\cos\theta,\ \ dxdydz \to r^2\sin\theta\, drd\theta d\varphi$

so that $V = 8\int_{r=0}^{R}\int_{\theta=0}^{\pi/2}\int_{\varphi=0}^{\pi/2} r^2\sin\theta\, d\varphi d\theta dr$

$$= 8\int_{r=0}^{R}\int_{\theta=0}^{\pi/2} r^2\sin\theta\frac{\pi}{2}\, dr d\theta$$

$$= 8\int_{r=0}^{R} r^2\frac{\pi}{2}\, dr = 8\frac{\pi}{6}R^3 = \frac{4}{3}\pi R^3$$

8. Moment of Inertia of a Cylinder about its Centre



– about x-axis

$$dI = R_x^2\rho dV$$

Use cylindrical polars *respecting the symmetry* of the object

$x \to r\cos\theta,\ \ y \to r\sin\theta,\ z \to z,\ dV \to r\, dr\, d\theta\, dz$

$$R_x^2 = y^2 + z^2 \to r^2\sin^2\theta + z^2 \qquad \rho = \frac{M}{2\pi a^2 b}$$

$$I = \rho\int_{z=-b}^{b}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} R_x^2 r dr d\theta dz = M\left(\tfrac{1}{4}a^2 + \tfrac{1}{3}b^2\right)$$