7. Div, Grad and Curl in Physics

7.0 ***Div, grad and curl often provide simple and elegant expressions in physics.***

7.1 ***Electrostatics:***

$\mathbf{E}$ = grad V

7.2 ***Electrodynamics:***

- **Coulomb's Law – Inverse Square Law**

  ⇒ Electric field out of volume proportional to charge enclosed

  $$\int_S \mathbf{E}.d\mathbf{S} = \frac{Q}{\varepsilon_0}$$

  Apply Stoke's Theorem, $\int_S \mathbf{E}.d\mathbf{S} = \int_V div\mathbf{E}.dV$

  Elemental volume has charge ρdV

  So div $\mathbf{E} = \rho/\varepsilon_0$

  In the absence of charge: div$\mathbf{E}$ = 0

  Maxwell's 1st Equation

- **Biot-Savart Law – Inverse Square Law**

  for Magnetic Field.

  But there is no magnetic charge

  (No magnetic monopoles)

  So div$\mathbf{B}$ = 0

  Maxwell's 2nd Equation

- **Faraday's Law – Induced e.m.f.**

  $$\oint \mathbf{E}.d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B}.d\mathbf{S}$$

  L.H.S. is just $\int_S \text{curl}\mathbf{E}.d\mathbf{S}$

  and so $\text{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

  Maxwell's 3rd Equation

- **Ampère's Law:**

$$\oint \mathbf{B}.d\mathbf{l} = \mu_0 I$$

(I is current enclosed by loop)

L.H.S. is just $\int_S \text{curl}\mathbf{B}.d\mathbf{S}$; R.H.S. is just $\mu_0 \int_S \mathbf{j}.d\mathbf{S}$

($\mathbf{j}$ is current density)

and so $\text{curl}\mathbf{B} = \mu_0 \mathbf{j}$

Half of Maxwell's 4^{th} Equation

For the other half, recall that

div curl = 0, so that

div curl **B** = div (μ_0 **j**) + **X**

Not 0 *therefore not* 0 *either*

So $\mathbf{X} = -\mu_0 \,\text{div}\, \mathbf{j} = \mu_0 \dfrac{\partial \rho}{\partial t} = \mu_0 \dfrac{\partial}{\partial t}\left(\varepsilon_0 \text{div}\mathbf{E}\right) = \text{div}\, \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$

And then $\text{curl}\mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$

Maxwell's 4^{th} Equation

7.3 *Maxwell's Equations in Free Space:*

1. $\nabla.\mathbf{E} = 0$
2. $\nabla.\mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
4. $\nabla \times \mathbf{B} = -\varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$

7.4 *The Electromagnetic Wave:*

Take curl of (3):

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t}$$

↓ identity ↓ order of differentiation

$$\nabla(\nabla.\mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mathbf{B}$$

$= 0$ from (1) $= \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$ from (4)

So $\nabla^2 \mathbf{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$ — WAVE EQUATION