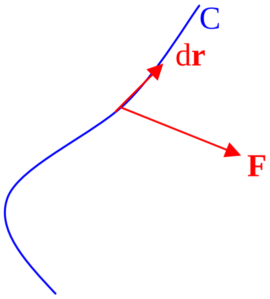
6. Line and Surface Integrals

6.0 *Integrals over Specified Paths and Surfaces*

6.1 *Example of a Line Integral:*

The work dW done by a force **F** when its point of application moves along a curve C through the displacement d**r** is:



$$dW = \mathbf{F}.d\mathbf{r}$$

So total work done is

$$W = \int_C \mathbf{F}.d\mathbf{r}$$

e.g. suppose that C arises from $x = t$, $y = 1/t$,

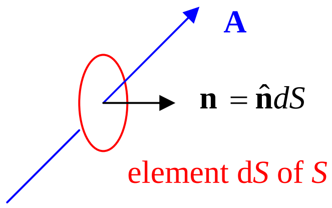
with $\mathbf{F} = (1, 1, 0)$, during $t = 1$ to 2.

$$d\mathbf{r} = (dx, dy) = (dt, -t^{-2}dt)$$

$$W = \int_C \mathbf{F}.d\mathbf{r} = \int_{t=1}^{2}\left(1 - \frac{1}{t^2}\right)dt = \underline{\underline{\tfrac{1}{2}}}$$

6.1 *Example of a Surface Integral:*

The flux Φ of a vector field **A** through a surface S with normal **n**



$$d\Phi = \mathbf{A}.\mathbf{n} = \mathbf{A}.\hat{\mathbf{n}}dS$$

Total Flux is

$$\Phi = \int_S \mathbf{A}.\mathbf{n} = \int_S \mathbf{A}.\hat{\mathbf{n}}dS$$

e.g. $\mathbf{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} = \dfrac{Q}{4\pi\varepsilon_0 r^3}\mathbf{r}$ and let Q be at centre of sphere S with radius a. Then $\hat{\mathbf{n}} = \hat{\mathbf{r}}$

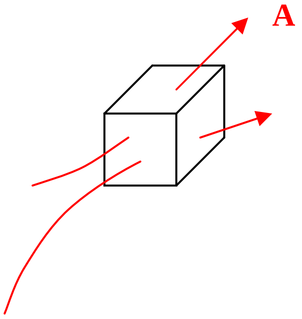
$$\text{So } \Phi = \int_S \frac{Q}{4\pi\varepsilon_0 a^2}dS = \frac{Q}{4\pi\varepsilon_0 a^2}\int_S dS = \frac{Q}{4\pi\varepsilon_0 a^2}4\pi a^2 = \underline{\underline{\frac{Q}{\varepsilon_0}}}$$

6.1 *Stoke's Theorems*

Volume Integral $\leftrightarrow$ Surface Integral

Area Integral $\leftrightarrow$ Line Integral

- **V $\leftrightarrow$ S**



Flux out of element dxdydz

is div $\mathbf{A}$ dx dy dz

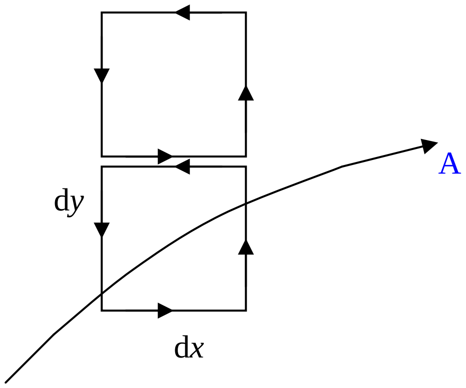
Flux out of collection of elements

is $\sum_i \text{div}\mathbf{A}_i dV_i$

So Flux out of a Volume (bounded by surface S) is

$$\Phi = \int_S \mathbf{A}.d\mathbf{S} = \int_V \text{div}\,\mathbf{A}\; dV$$

- **A $\leftrightarrow$ L**



$$curl\mathbf{A}dxdy = \sum_{i=1}^{4} \mathbf{A}.d\mathbf{l}_i$$

So $\oint \mathbf{A}.d\mathbf{l} = \int_A curl\mathbf{A}dA$