Fourier Analysis

Recall that a vector $\vec{v}$ has components, that $\vec{v} = (v_x, v_y, v_z)$

We find the components by 'projecting them out' using the projection operators $\hat{\imath}\cdot$, $\hat{\jmath}\cdot$, $\hat{k}\cdot$:

$$v_x = \hat{\imath}\cdot\vec{v} \quad v_y = \hat{\jmath}\cdot\vec{v} \quad v_z = \hat{k}\cdot\vec{v}$$

Fourier analysis is the same thing for functions.

Example: A musical note or chord.

What pure tones make it up?

(Overtones or harmonics?

Notes in chord?)

Fourier analysis can be done on periodic and non-periodic functions.

<u>Definition</u>: A function $f(x)$ is periodic with period L if

$$f(x+L) = f(x)$$

<u>Examples</u>: $\sin x$ has period 2π

$\cos 6x$ has period $\pi/3$

<u>Note</u>: If $f(x)$ has period L,

then $f(u) = f\left(\frac{Lx}{2\pi}\right)$ has period 2π

so we can restrict ourselves to functions of period 2π.

$\cos nx$, $\sin nx$, n interger, are periodic over 2π.

Essential Integrals:

$$\int_0^{2\pi} \cos mx \cos nx \, dx = \pi \delta_{mn}$$

$$\int_0^{2\pi} \sin mx \sin nx \, dx = \pi \delta_{mn}$$

$$\int_0^{2\pi} \cos mx \sin nx \, dx = 0$$

These integrals look like

$\hat{i} \cdot \hat{i} = 1 \qquad \hat{i} \cdot \hat{j} = 0$ etc

We use $\cos nx$, $\sin nx$ as basis

and $\int_0^{2\pi} dx$ as inner product.

Fourier coefficients:

$$a_k = \frac{1}{\pi} \int_0^{2\pi} \cos kx \, f(x) \, dx \qquad k = 0, 1, \ldots \infty$$

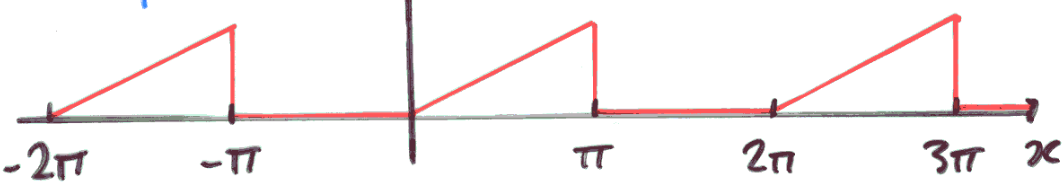
$$b_k = \frac{1}{\pi} \int_0^{2\pi} \sin kx \, f(x) \, dx \qquad k = 1, 2 \ldots \infty$$

FOURIER EXPANSION:

$$f(x) = \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

a_0, a_k, b_k are the Fourier coefficients of $f(x)$.

Example:



$$f(x) = \begin{cases} 0 & -\pi < x \leq 0 \\ x & 0 < x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x) = \frac{\pi}{2}$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos x \, dx = \frac{1}{\pi k^2}\left((-1)^k - 1\right)$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin x \, dx = -\frac{(-1)^k}{k}$$

$$f(x) = \frac{\pi}{4} + \frac{-2}{\pi}\cos x + \frac{1}{k}\sin x + 0 \cos 2x \ldots$$

Digression: Constructing Series

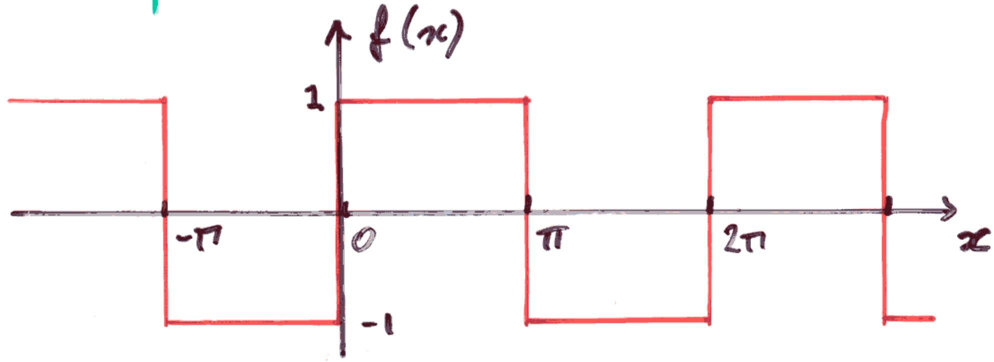
$$f(0) = 0 = a_0 + \sum_{k=1}^{\infty} a_k$$

$$= \frac{\pi}{4} + \sum_{k=1}^{\infty} -\frac{1}{k^2\pi}\left(1-(-1)^k\right)$$

$$= \frac{\pi}{4} + \frac{1}{\pi}\sum \frac{1}{(2k-1)^2}$$

i.e. $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{4}$$

Example: SQUARE WAVE



$$f(x) = \begin{cases} -1 & -\pi \leqslant x \leqslant 0 \\ +1 & 0 \leqslant x \leqslant \pi \end{cases}$$

$$a_0 = 0 \qquad a_k = 0 \qquad b_k = \frac{2}{k\pi}\left(1-(-1)^k\right)$$

Square Wave: $b_k = \frac{2}{k\pi}\left(1 - (-1)^k\right)$

$$= \frac{2}{\pi}\left(1, 0, -\frac{1}{3}, 0, +\frac{1}{5}, 0 \ldots\right)$$

Tidier to put $l = 2k$

Then $f(x) = \frac{4}{\pi} \sum_{l=1}^{\infty} \frac{\sin(2l-1)x}{2l-1}$

Digression: At $x = \frac{\pi}{2}$

$$f(x) = 1 = \frac{4}{\pi} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{2l-1}$$

i.e. $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots$

Square wave $f(x) = \sin x \cancel{+} - \frac{1}{3}\sin 3x \ldots$

Plotting successive terms...

In[2]:=

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Plot[Sin[x],{x,-Pi,Pi}]
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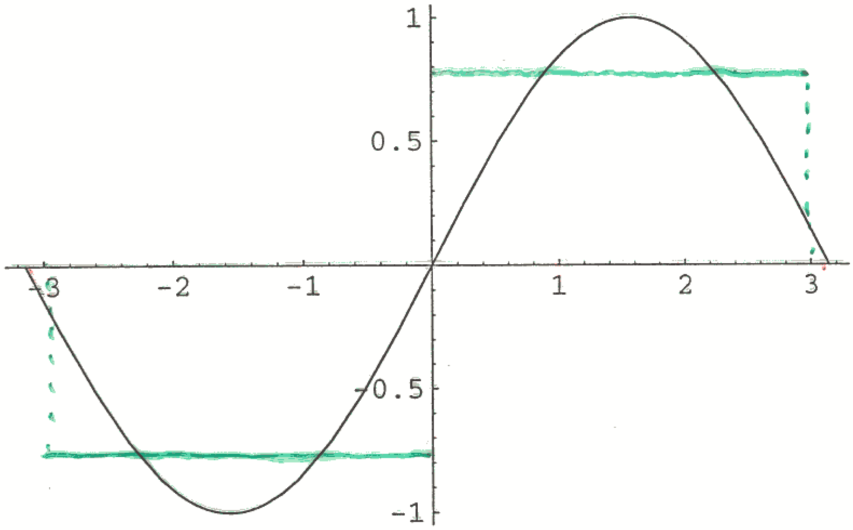

In[3]:=

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Plot[Sin[x]+Sin[3x]/3,{x,-Pi,Pi}]
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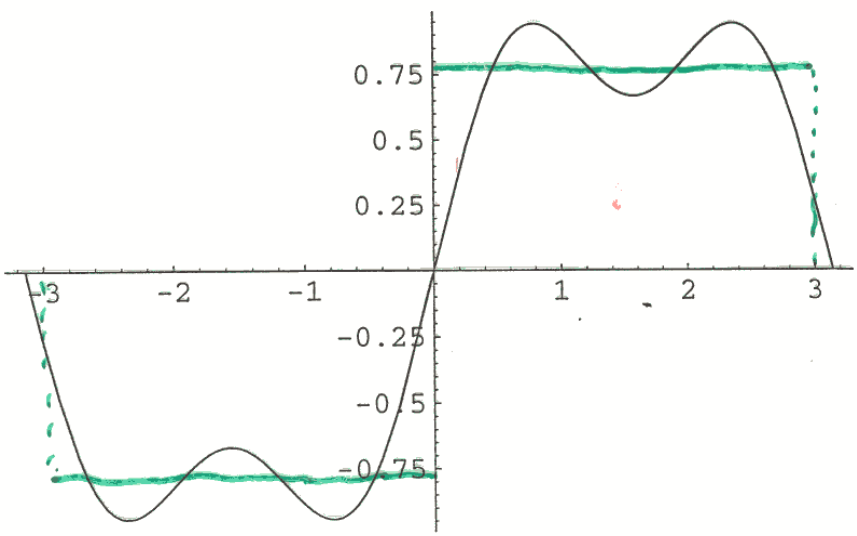

In[4]:=

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Plot[Sin[x]+Sin[3x]/3+Sin[5x]/5,{x,-Pi,Pi}]
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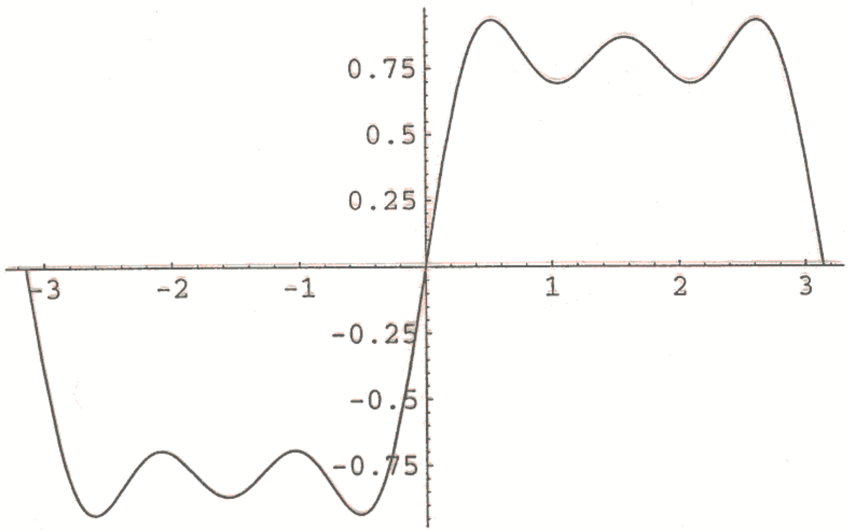

In[5]:=

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Plot[Sin[x]+Sin[3x]/3+Sin[5x]/5+Sin[7x]/7,{x,-Pi,Pi}]
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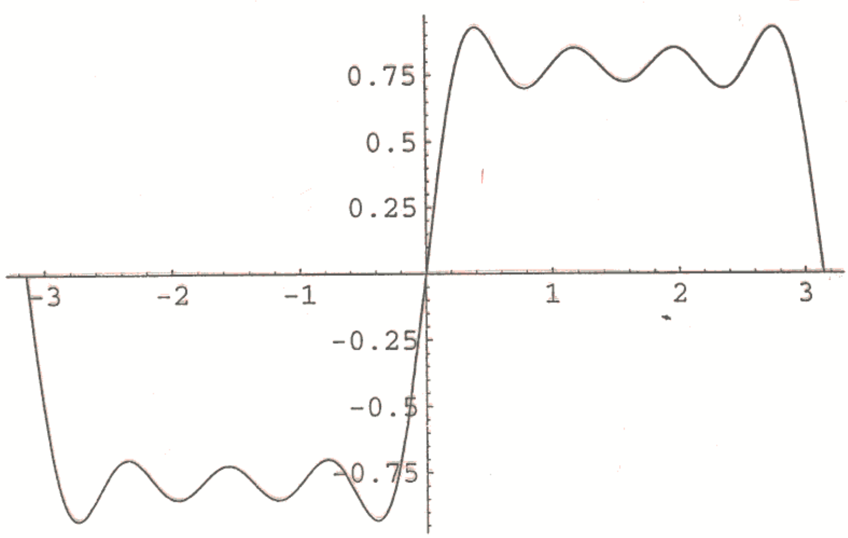

In[6]:=

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Plot[Sin[x]+Sin[3x]/3+Sin[5x]/5+Sin[7x]/7+Sin[9x]/9,{x,-Pi,
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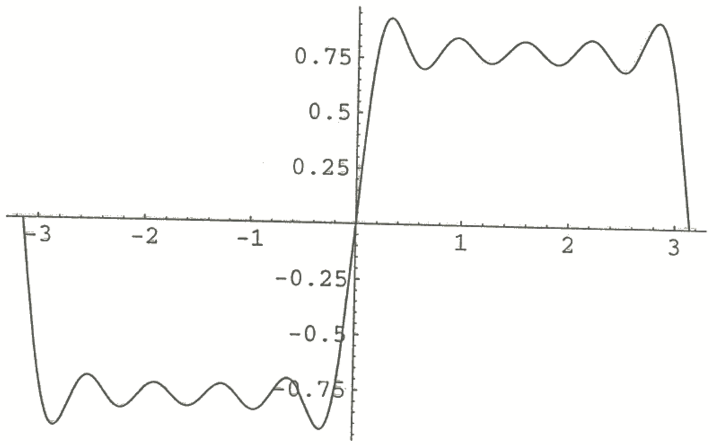

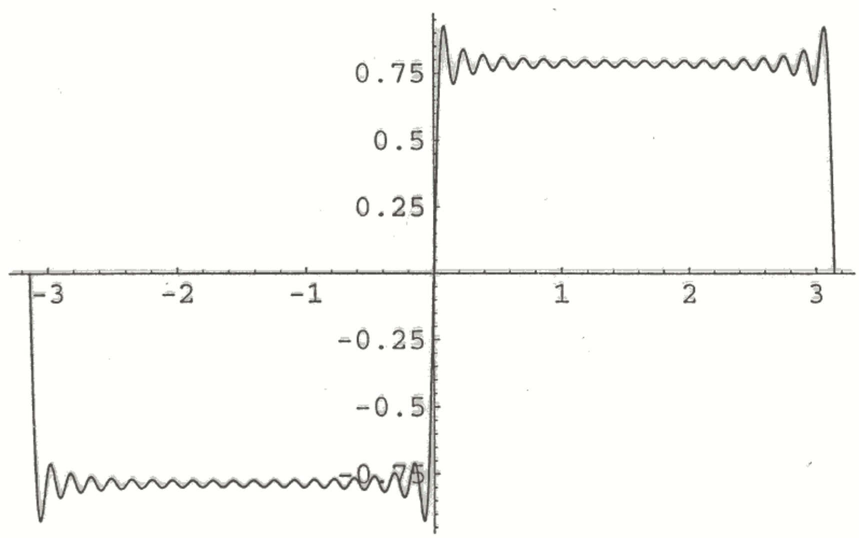

EVEN and ODD functions

Definitions: $f(x)$ is even if $f(-x) = f(x)$

$f(x)$ is odd if $f(-x) = -f(x$

for all x.

Examples: $\cos x$ is even

$\sin x$ is odd

Any function $f(x)$ can be written as the sum of an even and an odd function:

$$f(x) = \underbrace{\tfrac{1}{2}\big(f(x) + f(-x)\big)}_{\text{even}} + \underbrace{\tfrac{1}{2}\big(f(x) - f(-x)\big)}_{\text{odd}}$$

Fourier expansions:

$f(x)$ even, then all $b_k = 0$

$f(x)$ odd, then all $a_k = 0$

COMPLEX representation of Fourier series for real functions.

We have $e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x$

so $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$

$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

Fourier series is

$$f(x) = \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$= \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} \tfrac{1}{2}a_k e^{+ikx} - \tfrac{i}{2} b_k e^{+ikx}$$

$$+ \tfrac{1}{2}a_k e^{-ikx} + \tfrac{i}{2} b_k e^{-ikx}$$

$$= \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} c_k e^{ikx} + c_k^* e^{-ikx}$$

with $c = \frac{1}{2}(a_k - ib_k)$ $c^* = \frac{1}{2}(a_k + ib_k)$

Now let $c_{-k} = c_k^*$

Then $\underline{\underline{f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}}}$

For complex functions, we have

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad \text{without } c_{-k} = c_k^* .$$

Finding c_k: Consider the integral

$$\int_0^{2\pi} f(x) e^{-ik'x} dx = \int_0^{2\pi} \sum_{k=-\infty}^{\infty} c_k e^{i(k-k')x} dx$$

$$= \sum_{k=-\infty}^{\infty} c_k \underbrace{\int_0^{2\pi} e^{i(k-k')x} dx}$$

$$\int_0^{2\pi} e^{ikx} dx = 2\pi \delta_{k,0} = \begin{cases} 2\pi & k=0 \\ 0 & k \neq 0 \end{cases}$$

$$= \sum_{k=-\infty}^{\infty} c_k \delta_{k-k',0} 2\pi = \underline{\underline{c_{k'}}}$$

So $$\underline{\underline{c_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx}}$$

NON-PERIODIC FUNCTIONS:

The FOURIER TRANSFORM

Aperiodic functions $f(t)$ may have some of any frequency ω, so instead of

$$c_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\, e^{-ikt}\, dt \qquad k \text{ integer}$$

we let k take any value. Then

$$F(k) = \int_{-\infty}^{\infty} f(t)\, e^{+ikt}\, dt$$

Re-assembly of $f(t)$:

$$f(t) = \sum_{k=-\infty}^{+\infty} c_k\, e^{ikt}$$

becomes:

$$f(t) = \int_{k=-\infty}^{\infty} F(k)\, e^{-ikt}\, dk$$

Proof and Prefactor

$$F(k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k') e^{-ik't} dk' e^{+ikt} dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k') e^{i(k-k')t} dt \, dk'$$

$$= \int_{-\infty}^{\infty} F(k') \underbrace{\int_{-\infty}^{\infty} e^{i(k-k')t} dt}_{\substack{\text{Zero by oscillation} \\ \text{for } k \neq k' \\ 2\pi \text{ by definition} \\ \text{for } k = k'}} dk'$$

$$= 2\pi F(k)$$

So we spread the 2π out over the forward and back transforms:

Forward Fourier Transform:

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ikt} dt$$

Back Fourier Transform:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{-ikt} dk$$