9. Determinants

9.0 THE SIZE OF A MATRIX.

For a square matrix **A**, the determinant *det*(**A**) = |**A**| is a scalar quantity, the 'size' of **A.**

9.1 Recursive definition:

(i) $n = 1$, **e.g.** $\mathbf{A} = (a)$, $det(\mathbf{A}) = a$

(ii) $n = 2$, **e.g.** $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $det(\mathbf{A}) = ad - bc$

Examples: $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$, $\quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$,

$$\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

(iii)_$n = 3$, **e.g.** $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$,

$$det(\mathbf{A}) = a\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} - b\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + c\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

o Minors

This expression for the determinant of **A** is equivalent to

$$det(\mathbf{A}) = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

where the *minor* M_{ij} of the element a_{ij} is the determinant of the matrix formed by deleting the i^{th} row and the j^{th} column from **A**.

(iv)_$n \times n$, $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix}$

Then $\det(\mathbf{A}) = (-1)^i \sum_{j=1}^{n} (-1)^j a_{ij} M_{ij}$

det of $n \times n$ *det of* $(n-1) \times (n-1)$

where M_{ij} is the *minor* of a_{ij} as above.

- Note: i can be any number 1 to n, not just 1 as in the examples.

Examples:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0 \times \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + 0 \times \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 1 \times 1 + 0 \times 0 + 0 \times 0 = \underline{\underline{1}}$$

$$\begin{vmatrix} 1 & 7 & 5 \\ -4 & 4 & 8 \\ 2 & 6 & 9 \end{vmatrix} = 1 \times \begin{vmatrix} 4 & 8 \\ 6 & 9 \end{vmatrix} - 7 \times \begin{vmatrix} -4 & 8 \\ 2 & 9 \end{vmatrix} + 5 \times \begin{vmatrix} -4 & 4 \\ 2 & 6 \end{vmatrix}$$

$$= 1 \times (36 - 48) - 7(-36 - 16) + 5 \times (-24 - 8) = \underline{\underline{192}}$$

$$\begin{vmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{vmatrix} = a \times \begin{vmatrix} c & 0 \\ 0 & e \end{vmatrix} - 0 \times \begin{vmatrix} 0 & 0 \\ d & e \end{vmatrix} + b \times \begin{vmatrix} 0 & c \\ d & 0 \end{vmatrix} = \underline{\underline{ace - bcd}}$$

Example using Minors:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$M_{11} = 5 \times 9 - 6 \times 8 = -3$
$M_{12} = 4 \times 9 - 6 \times 7 = -6$
$M_{13} = 4 \times 8 - 5 \times 7 = -3$

$$\begin{aligned} \det(\mathbf{A}) &= (-1)^1 \{(-1)^1 \times 1 \times -3 + (-1)^2 \times 2 \times -6 + (-1)^3 \times 3 \times -3\} \\ &= -1 \times (3 - 12 + 9) \\ &= 0 \end{aligned}$$

OR

$M_{21} = 2 \times 9 - 3 \times 8 = -6$
$M_{22} = 1 \times 9 - 3 \times 7 = -12$
$M_{23} = 1 \times 8 - 2 \times 7 = -6$

$$\begin{aligned} \det(\mathbf{A}) &= (-1)^1 \{(-1)^1 \times 4 \times -6 + (-1)^2 \times 5 \times -12 + (-1)^3 \times 6 \times -6\} \\ &= -1 \times (24 - 60 + 36) \\ &= 0 \end{aligned}$$

o Cofactors of Matrix Elements

To tidy up the recursive definition of the determinant,

Let the cofactor A_{ij} of the element a_{ij} of the matrix **A** be

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Then $\det(A) = \sum_{j=1}^{n} a_{ij} M_{ij}$ for any i.

9.2. Direct Definition of Determinants

(i) Define the symbol $\varepsilon_{i_1 i_2 i_3 \cdots i_n}$ for $i_1, i_2, i_3, \ldots i_n$, being numbers between 1 and n.

(ii) Let $$\varepsilon_{i_1 i_2 i_3 \cdots i_n} = \begin{cases} +1 & \text{if } (i_1 \cdots i_n) \text{ is an even permutation of } (1 \cdots n) \\ -1 & \text{if } (i_1 \cdots i_n) \text{ is an odd permutation of } (1 \cdots n) \\ 0 & \text{otherwise (if any of the } i\text{'s are equal)} \end{cases}$$

Examples: $\varepsilon_{12} = +1$, $\varepsilon_{21} = -1$, $\varepsilon_{11} = 0$, $\varepsilon_{22} = 0$.

$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$,

$\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$,

$\varepsilon_{223} = \varepsilon_{331} =$ **another 19 = 0.**

(iii) Then

$$\det(A) = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \sum_{\text{All choices}} \varepsilon_{i_1 i_2 i_3 \cdots i_n} \times a_{1i_1} \times a_{2i_2} \cdots \times a_{ni_n}$$

The determinant is thus the sum of n^n terms, each the product of n elements of **A**. **All but $n!$ are put to zero by the factors ε.**

Note: $\det(\mathbf{A}^T) = \det(\mathbf{A})$

$\det(\lambda\mathbf{A}) = \lambda^n \det(\mathbf{A})$

$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det(\mathbf{A}) \times \det(\mathbf{B})$

9.3 MORE MATRIX DEFINITIONS.

Note: $det(\mathrm{E}) = 1$

- Orthogonal Matrices

An $n \times n$ matrix **O** is orthogonal if

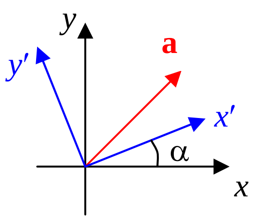
$$\mathbf{O}^T\mathbf{O} = \mathbf{O}\mathbf{O}^T = \mathbf{E}$$

i.e. the transpose is the inverse, equivalently, $\mathbf{O}^T = \mathbf{O}^{-1}$

So, $det(\mathrm{O})\, det(\mathrm{O}^T) = 1$, and then $det(\mathrm{O}) = \pm 1$

ROTATION MATRICES in *n-dimensional space* are $n \times n$ Orthogonal Matrices.

Example:



$\mathbf{a} = (a_x, a_y) = (a'_x, a'_y)$

$a'_x = a_x \cos\alpha + a_y \sin\alpha$
$a'_y = -a_x \sin\alpha + a_y \cos\alpha$

$$\text{i.e.} \begin{pmatrix} a'_x \\ a'_y \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

Matrix $\mathbf{R}_\alpha$

$$R_\alpha^T = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$R_\alpha^T R_\alpha = \begin{pmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha - \cos\alpha\sin\alpha & \cos^2\alpha + \sin^2\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E$$

So, $\mathbf{R}^T = \mathbf{R}^{-1} = \mathbf{R}_{-\alpha}$

- Unitary Matrices

An $n \times n$ matrix **U** is unitary if

$$\mathbf{U}^{\dagger}\mathbf{U} = \mathbf{U}\mathbf{U}^{\dagger} = \mathbf{E}$$

(**Recall** $\mathbf{U}^{\dagger} = (\mathbf{U}^{T})^{*} = (\mathbf{U}^{*})^{T}$)

i.e. the Hermitian conjugate is the inverse, equivalently, $\mathbf{U}^{\dagger} = \mathbf{U}^{-1}$

So, $det(\mathrm{U})\, det(\mathrm{U}^{\dagger}) = 1$**, and then** $det(\mathrm{U}) = \pm 1$

Example:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}$$

$$U^{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix} \qquad (U^{T})^{*} = U^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}$$

$$U^{\dagger}U = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E$$