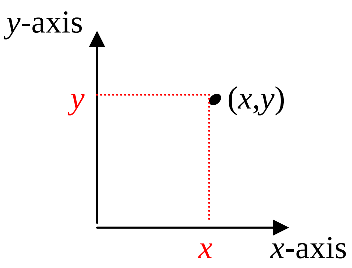
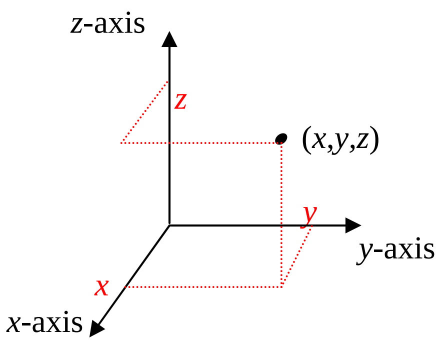
2. Coordinates and Integration

2.0 *Cartesian (rectilinear) coordinates.*

Two-dimensional

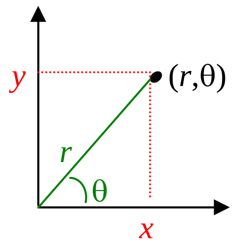


Three-dimensional



These are very common, in 1, 2, 3, 4 . . . n . . . ∞ dimensions, but, as with complex numbers, other representations are often useful:

2.1 *Polar coordinates*



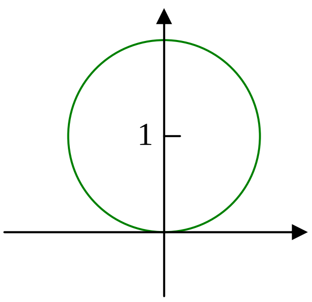
$x = r \cos \theta$
$y = r \sin \theta$
$r = \mathrm{sqrt}(x^2 + y^2)$
$\theta = \arctan(y/x)$

Functions in Polar Coordinates:

A curve expressed as a function:

$r = f(\theta)$ [compare $y = f(x)$]

E.g. $r = 2 \sin \theta$



(See Stroud 22.8)

A function of position:

$f(r, \theta)$ [compare $f(x, y)$]

Conversions:

$$\frac{\partial f(r,\theta)}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}$$

and

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r} \qquad \frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{y}{r^2}$$

so

$$\frac{\partial f(r,\theta)}{\partial x} = \frac{x}{r}\frac{\partial f}{\partial r} - \frac{y}{r^2}\frac{\partial f}{\partial \theta} = \cos\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial \theta}$$

Similarly,

$$\frac{\partial f(x,y)}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \ldots$$

$$\frac{\partial f(x,y)}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \ldots \qquad \textit{Exercise}$$

Full Differentials:

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta$$

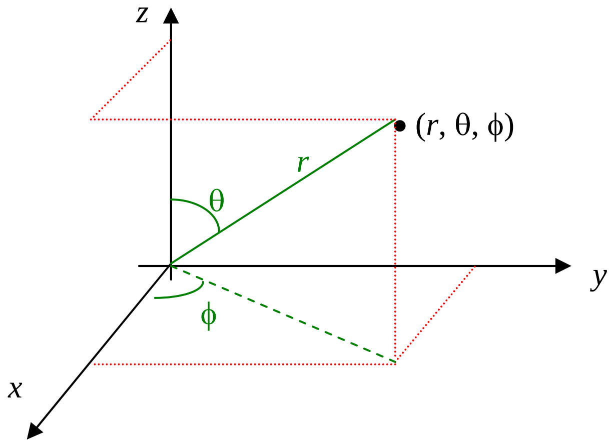
$$dr = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy$$

$$d\theta = \frac{\partial \theta}{\partial x}dx + \frac{\partial \theta}{\partial y}dy$$

Double Integrals:

$$\iint dx\,dy = \iint r\,dr\,d\theta$$

2.2 *Spherical Polar coordinates*



Ranges: $0 \le r < \infty$
$0 \le \theta < \pi$
$0 \le \phi < 2\pi$

Conversions:

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

$r = \mathrm{sqrt}(x^2 + y^2 + z^2)$

$\theta = \cos^{-1} z / \mathrm{sqrt}(x^2 + y^2 + z^2)$

$\phi = \tan^{-1} y / x$

Full Differentials:

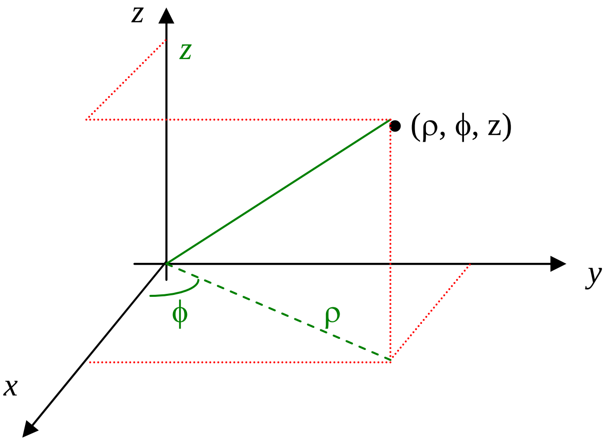
$$dr = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy + \frac{\partial r}{\partial z}dz$$

Etc, for $d\theta$, $d\phi$, dx, dy, dz.

Triple Integrals:

$$\int_V dV = \iiint dxdydz = \iiint r^2 \sin\theta\, dr d\theta d\phi$$

2.2 *Cylindrical Polar coordinates*



Ranges: $0 \le \rho < \infty$

$0 \le \phi < 2\pi$

$-\infty \le z < \infty$

Conversions:

$x = \rho \cos \phi$	$\rho = \mathrm{sqrt}(x^2 + y^2)$
$y = \rho \sin \phi$	$\theta = \tan^{-1} y / x$
$z = z$	$z = z$

Full Differentials: Exercise

Triple Integrals: Exercise