MT2 Course Outline

Week	Exam	Topic	Homework
1	1st	Complex Numbers	
2	1st	Coordinates and Integration . . .	HW1
3	1st	. . . cont'd, Vectors . . .	HW2
4	1st	. . . cont'd, Vector Fields	HW3
5	1st	Line and Surface Integrals, Div, Grad and Curl in Physics	HW4
6	2nd	Matrices	HW5
7		Reading Week (no classes)	
8	2nd	FIRST EXAM. Matrices, Determinants	HW6
9	2nd	. . . cont'd, Eigenvectors	HW7
10	2nd	. . . cont'd, Fourier Analysis, Differential Equations . . .	HW8
11	2nd	. . . cont'd, . . .	HW9
12	2nd	. . . cont'd. SECOND EXAM.	

1. Revision of Complex Numbers

1.1 *Definitions and Rules*

- $\underline{c = a + ib}$

  '*complex*' = '*real*' + $i\times$'*real*' $i = +\sqrt{-1}$

  $a = \text{Re}(c)$ real part of c

  $b = \text{Im}(c)$ imaginary part of c

- Addition: Let $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$

  Then $c_1 + c_2 = a_1 + ib_1 + a_2 + ib_2 = a_1 + a_2 + ib_1 + ib_2$

  $\underline{= (a_1 + a_2) + i(b_1 + b_2)}$

- Subtraction: Let $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$

  Then $c_1 - c_2 = a_1 + ib_1 - a_2 - ib_2 = a_1 - a_2 + ib_1 - ib_2$

  $\underline{= (a_1 - a_2) - i(b_1 + b_2)}$

- Multiplication: Let $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$

  Then $c_1\, c_2 = a_1a_2 + a_1ib_2 + ib_1a_2 + ib_1ib_2 = a_1a_2 + a_1ib_2 + ib_1a_2 + i^2b_1b_2$

  $\underline{= (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)}$

- Complex Conjugate: Let $c = a + ib$

  Its '*complex conjugate*' is $\underline{c^* = a - ib}$

  Notice that

  $cc^* = (a + ib)\,(a - ib) = a^2 + b^2$

  which is real

- <u>Modulus:</u> Let $c = a + ib$

  Its '*modulus*' is $|c| = \sqrt{cc^*} = \sqrt{a^2 + b^2}$

  Notice that

  $|c| \geq 0$ *and that* $|c| = 0$ *if and only if* $c^* = 0$, i.e. $a = 0$ and $b = 0$.

- <u>Division:</u> Let $c_1 = a + ib$ and $c_2 = x + iy$

  Then $\frac{c_1}{c_2} = \frac{a+ib}{x+iy} = \frac{(a+ib)(x-iy)}{(x+iy)(x-iy)} = \frac{(ax-by)+i(bx-ay)}{x^2+y^2}$

  $= \frac{ax-by}{x^2+y^2} + i\frac{bx-ay}{x^2+y^2}$

  = real + i × real = complex number

- <u>Solutions of Equations:</u> E.g. $ax^2 + bx + c = 0$

  Solutions are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

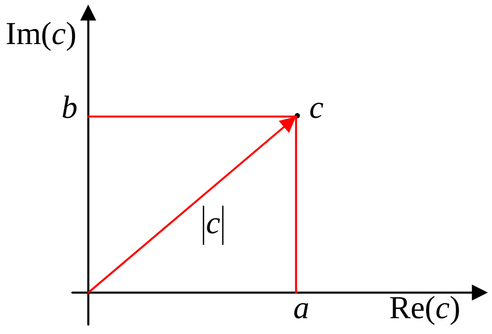
  If b^2-$4ac$ ρ 0 then solutions are real

  If b^2-$4ac < 0$ then solutions are complex, and

  $$x = -\frac{b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}$$

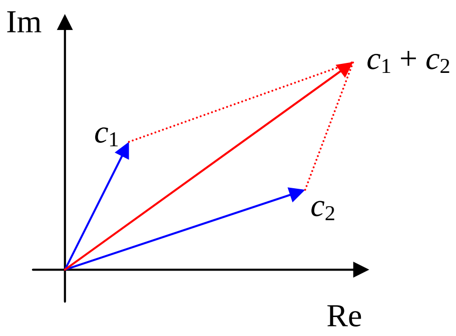
  real + i × real

1.2 *The Argand Diagram*



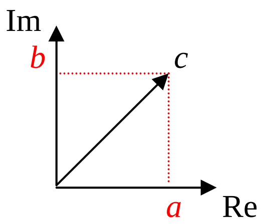
E.g. $a = 3$
$b = 4$
$c = 3 + 4i$
$|c| = 5$

Complex Numbers, *like Vectors*, *are "Ordered Pairs of Numbers"*, *and so add the same way*

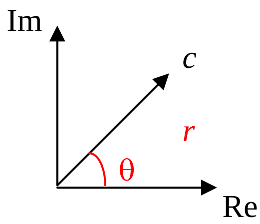


Note:
$c_1 - c_2 = c_1 + (-c_2)$
Subtract by adding $-c_2$ to c_1

1.3 *Polar Form of Complex Numbers*



is the same as



$a = r\cos\theta$

$b = r\sin\theta$

$r = \sqrt{a^2 + b^2} = |c|$

$\theta = \arg(c) = \tan^{-1}\dfrac{b}{a}$

Now, a well-known series is:

$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} + \ldots$$

So,

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2} + \ldots + \frac{(i\theta)^n}{n!} + \ldots$$

Collecting alternate terms,

$$e^{i\theta} = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} \ldots + \frac{(-1)^{2n}\theta^{2n}}{(2n)!} + \ldots$$

$$+ i\left(\theta - \frac{\theta^3}{3!} + \ldots + \frac{(-1)^{2n+1}\theta^{2n+1}}{(2n+1)!} + \ldots\right)$$

Which can be identified as the series expansion of

$e^{i\theta} = \cos\theta + i\sin\theta$

So we have

$c = a + ib = r(\cos\theta + i\sin\theta) = r\,e^{i\theta}$

Polar Form

This is useful for

- Multiplication and Division

  $$c_1 c_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

  $$\frac{c_1}{c_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

- de Moivre's Theorem

  $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

  Proof: $(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n$

  $= e^{i(n\theta)}$

  $= \cos n\theta + i\sin n\theta$

- One of the most beautiful relationships in mathematics:

  Put $\theta = \pi$ (180°) in de Moivre's theorem. Then

  $$e^{i\pi} + 1 = 0$$

1.4 *Complex Roots*

First notice that: $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$

So $e^{2\pi n i} = 1$ also (with n integer)

And $e^{i(2\pi n+\theta)} = e^{i\theta}$

Taking roots: For $c = re^{i\theta} = e^{i(2\pi k+\theta)}$

$$\sqrt[n]{c} = c^{1/n} = r^{1/n} e^{\frac{i(\theta+2\pi k)}{n}} \qquad \text{with } k = 0, 1, 2 \ldots n-1$$

are the n different n^{th} roots of c. (For $k \geq n$ they repeat.)

1.5 *Complex Variables*

Complex numbers used as *variables* in *functions*. We often use 'z' as a complex variable, and 'c' as a complex constant.

So $z = x + iy = re^{i\theta}$ with $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$, is a point anywhere on the Argand diagram, a point on the '*Complex Plane*'

Examples:

1. $|z|$ is a *function* of z. We may have

$$f(z) = |z| = R$$

which we can solve to get an equation for x and y

(the equation of a circle of radius R.)

2. $\mathrm{Arg}(z)$ is a function of z. We may have

$g(z) = \mathrm{Arg}(z) = \theta_0$

which we can solve to get an equation for x and y

(the equation of a straight line of gradient θ_0.)

3. $\left|\frac{z-c}{z+c}\right|^2 = 1$ Solve for x and y, getting

$ax + by = 0$ (a straight line) – Exercise: prove it

4. $\alpha\,(z^2 + z^{*2}) + 2\beta\, z\, z^* = 1$

with $\alpha = ¼\,(a^{-2} - b^{-2})$ and $\beta = ¼\,(a^{-2} + b^{-2})$

Simplifies to $x^2 a^{-2} + y^2 b^{-2} = 1$ – Ellipse

1.5.2 *Integrating Complex Functions*

Complex functions integrate just like real functions. Examples:

1. $\int_0^{2\pi} e^{ikx} dx = \int_0^{2\pi} (\cos kx + i \sin kx) dx = \left[\frac{\sin kx}{k} - i \frac{\cos kx}{k} \right]_0^{2\pi} = 0$

k integer, $k \neq 0$.

If $k = 0$ then $\int_0^{2\pi} e^{ikx} dx = \int_0^{2\pi} 1 dx = [x]_0^{2\pi} = 2\pi$

These results are used later, for *Fourier Series*. They may be stated as

$$\int_0^{2\pi} e^{ikx} dx = \begin{cases} 0 \text{ if } k \neq 0 \\ 2\pi \text{ if } k = 0 \end{cases} = 2\pi \delta_{0k}$$

The Kronecker delta is defined by

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

i, j integers

2. $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2 + ikx} dx$

Rearrange:

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-ik)^2 - \frac{1}{2}k^2} dx$$

Substitute u for $x - ik$

$$= e^{-\frac{1}{2}k^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du$$

$$= \sqrt{2\pi}\, e^{\frac{1}{2}k^2}$$

(which is needed in quantum mechanics.)

Exercise: Can you show that $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$