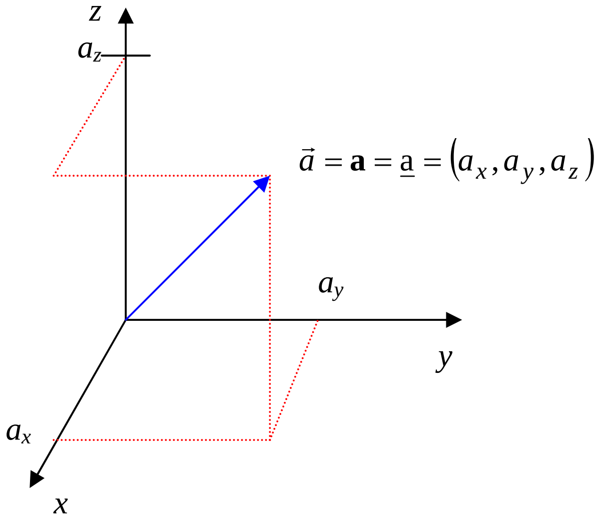
4. Vectors

4.0 ***Vectors are ordered multiplets of numbers. In three dimensions these are ordered triplets,*** $\vec{u} = (u_x, u_y, u_z)$.

The three numbers are the components of the vector, in a rectilinear co-ordinate system:



4.1 ***Definitions***

- Addition

$$\mathbf{u} = (u_x, u_y, u_z)$$
$$\mathbf{v} = (v_x, v_y, v_z)$$
$$\mathbf{w} = \mathbf{u} + \mathbf{v} = (u_x + v_x, u_y + v_y, u_z + v_z)$$

- Multiplication by a number λ

$$\mathbf{u} = (u_x, u_y, u_z)$$
$$\mathbf{w} = \lambda\mathbf{u} = (\lambda u_x, \lambda u_y, \lambda u_z)$$

Some Consequences:

$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	**Commutivity**
$\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$	**Distributivity**
$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$	**Associativity**
$\exists\ \mathbf{O} = (0, 0, 0)$	**Null Vector**
$\exists\ \hat{\mathbf{i}} = (1, 0, 0)$	**Unit . . .**
$\exists\ \hat{\mathbf{j}} = (0, 1, 0)$	**. . . Vectors along . . .**
$\exists\ \hat{\mathbf{k}} = (0, 0, 1)$	**. . . Co-ordinates**
$\lvert\mathbf{a}\rvert = \sqrt{a_x^2 + a_y^2 + a_z^2}$	**Modulus, or Length**

- Multiplication by a vector

  1. Vector 'times' vector $\rightarrow$ number
     $\mathbf{u.v} \rightarrow$ scalar

  2. Vector 'times' vector $\rightarrow$ vector
     $\mathbf{u}\times\mathbf{v} \rightarrow$ vector

- The Dot Product

$$\mathbf{u.v} = (u_x, u_y, u_z).(v_x, v_y, v_z)$$
$$= \sum_{i=1}^{3} u_i v_i = (u_x v_x + u_y v_y + u_z v_z)$$

Some Consequences:

$\mathbf{a.b} = \mathbf{b.a}$	**Commutivity**
$\mathbf{a.a} = \lvert\mathbf{a}\rvert^2$	**Square of length**
$\mathbf{a.(b + c) = a.b + a.c}$	**Distributivity**
$\mathbf{a.b} = \lvert\mathbf{a}\rvert \times \lvert\mathbf{b}\rvert \times \cos\theta$	**θ is angle between a and b**
$\hat{\mathbf{i}}.\mathbf{a} = (\mathbf{1, 0, 0}).(a_x, a_y, a_z) = a_x$	**i.** ***projects out*** a_x
$\hat{\mathbf{j}}.\mathbf{a} = (\mathbf{0, 1, 0}).(a_x, a_y, a_z) = a_y$	**j.** ***projects out*** a_y
$\hat{\mathbf{k}}.\mathbf{a} = (\mathbf{0, 0, 1}).(a_x, a_y, a_z) = a_z$	**k.** ***projects out*** a_z
$\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$	**i, j and k** ***span the space***
$\hat{\mathbf{a}} = \frac{\mathbf{a}}{\lvert\mathbf{a}\rvert}$	**Unit vector along a**

- The Cross Product

$$\mathbf{u}\times\mathbf{v} = (u_x, u_y, u_z) \times (v_x, v_y, v_z)$$
$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = (u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x)$$

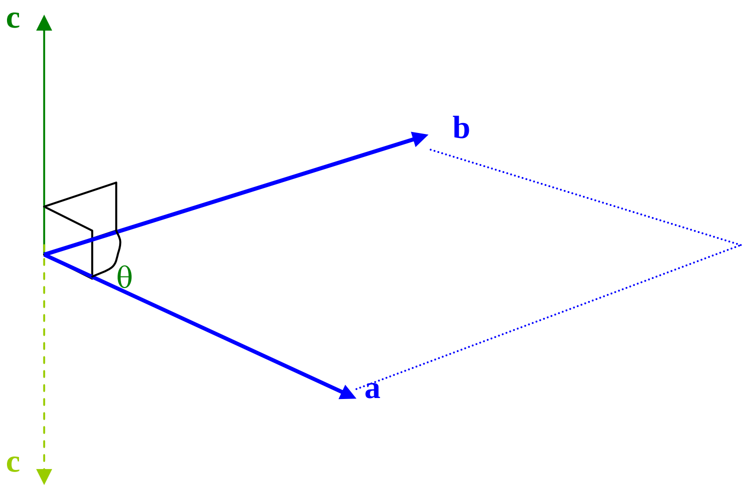
which is a vector, – normal to plane of **u** and **v**
– of length $\lvert\mathbf{u}\rvert \times \lvert\mathbf{v}\rvert \times \sin\theta$

Some Consequences:

$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$	**Non-Commutivity**
$\mathbf{a}\times(\mathbf{b} + \mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$	**Distributivity**
$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$	**Non-Associativity**

$$\hat{\mathbf{i}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}\times\hat{\mathbf{k}} = 0$$
$$\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}},\ \hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}},\ \hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}$$
$$\mathbf{a}.(\mathbf{a}\times\mathbf{b}) = 0 = \mathbf{b}.(\mathbf{a}\times\mathbf{b})$$

- Pseudovectors — The direction (±) of $\mathbf{a}\times\mathbf{b}$ is conventional:



We use the ***right-hand rule*** according to which a spiral, screw-thread, corkscrew, etc, *turning from* **a** *to* **b,** *advances along* +ve **c**

4.2 *Interpretation of Dot and Cross Products*

- Dot Product — **How alike** are two vectors?
  How much of one is in the other?

- Cross Product — **Area** of the parallelogram
  which is **spanned** by two vectors
  represented as a vector by its **normal**

- Physical Examples — **Flux** of a field **E** through a surface **S** is **E.S**
  Torque is **T = r×F**
  Angular momentum is **L = r×p**

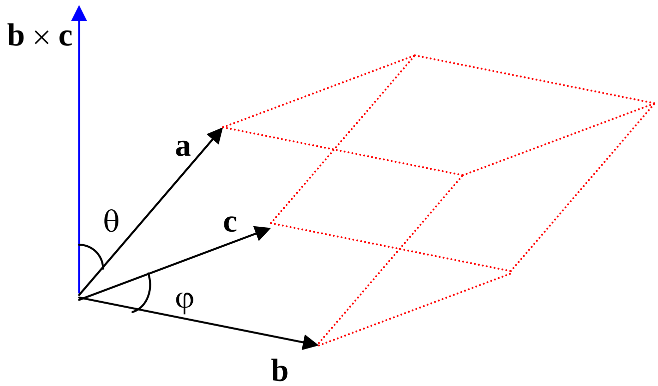
4.3 *Triple Products*

- $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$ is a number (a scalar)
  It is the volume of the parallelepiped (warped cube)
  spanned by **a**, **b** and **c**

Proof: $\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = |\mathbf{a}||\mathbf{b} \times \mathbf{c}| \cos\theta$

$= |\mathbf{a}| \cos\theta \; (|\mathbf{b}||\mathbf{c}| \sin\varphi$

Height *Area of base*



- *Coplanarity:* By inspection, if **a**, **b** and **c** are all non-zero, then
  $V = \mathbf{a}.(\mathbf{b} \times \mathbf{c}) = 0$ if and only if **a**, **b** and **c** are coplanar.

Formal Proofs:

(1) If **a**, **b**, **c** are coplanar,

Then $\mathbf{a} = \beta\mathbf{b} + \gamma\mathbf{c}$ for some β, γ

So $\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = (\beta\mathbf{b} + \gamma\mathbf{c}).(\mathbf{b} \times \mathbf{c})$

$= \beta\mathbf{b}.(\mathbf{b} \times \mathbf{c}) + \gamma\mathbf{c}.(\mathbf{b} \times \mathbf{c})$

$= 0$ by an earlier result.

Thus coplanarity $\Rightarrow \mathbf{a}.(\mathbf{b} \times \mathbf{c}) = 0$

(2) Let $\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = 0$ and assume linear independence

$(\mathbf{a} \neq \beta\mathbf{b} + \gamma\mathbf{c})$

i.e. $\mathbf{a} = \beta\mathbf{b} + \gamma\mathbf{c} + \delta\mathbf{d}$ for some δ and $\mathbf{d} \perp \mathbf{b}$, $\mathbf{d} \perp \mathbf{c}$

Then $0 = \mathbf{a}.(\mathbf{b} \times \mathbf{c}) = (\beta\mathbf{b} + \gamma\mathbf{c} + \delta\mathbf{d}).(\mathbf{b} \times \mathbf{c})$

$= \delta\mathbf{d}.(\mathbf{b} \times \mathbf{c}) \neq 0$ (because $(\mathbf{b} \times \mathbf{c})7\mathbf{d}$)

The assumption has generated a contradiction,
therefore must be false.

4.4 *Two Identities*

- **Lagrange's Identity**

  $$(\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d}) = (\mathbf{a}.\mathbf{c})\,(\mathbf{b}.\mathbf{d}) - (\mathbf{b}.\mathbf{c})\,(\mathbf{a}.\mathbf{d})$$

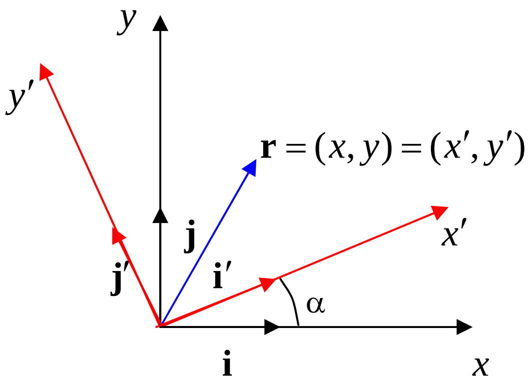
  Use Maple to prove this.

- **Another Identity**

  $$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{b}.\mathbf{c})\mathbf{a}$$

  Use Maple to prove this.

4.5 *Rotations of Coordinates*



x', y' are coordinate axes rotated by angle α w.r.t. x, y

$$\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{i}'.\mathbf{i}' = \mathbf{j}'.\mathbf{j}' = 1$$
$$\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{i} = \mathbf{i}'.\mathbf{j}' = \mathbf{j}'.\mathbf{i}' = 0$$
$$\mathbf{i}.\mathbf{i}' = \mathbf{j}.\mathbf{j}' = \cos\alpha$$
$$\mathbf{i}.\mathbf{j}' = \cos\left(\alpha + \tfrac{\pi}{2}\right) = -\sin\alpha$$
$$\mathbf{i}'.\mathbf{j} = \cos\left(-\alpha + \tfrac{\pi}{2}\right) = \sin\alpha$$
$$\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} = r'_x\mathbf{i}' + r'_y\mathbf{j}'$$

So,

$$r_x = \mathbf{r}.\mathbf{i} = (r'_x\mathbf{i}' + r'_y\mathbf{j}').\mathbf{i}$$
$$= r'_x \cos\alpha - r'_y \sin\alpha$$

Similarly,

$$r_y = r'_x \sin\alpha + r'_y \cos\alpha$$

Thus the coordinate transformation is

$$x = x'\cos\alpha - y'\sin\alpha$$
$$y = x'\sin\alpha + y'\cos\alpha$$

and equivalently

$$x' = x\cos\alpha + y\sin\alpha$$
$$y' = x\sin\alpha - y\cos\alpha$$

- **The formal definition** of a vector in two dimensions is:

  An ordered pair of numbers (x, y) that transform as above.

- **The formal definition** of a scalar is:

  A quantity (a number) x that is invariant under rotation.