6 Fourier Analysis

Have expanded arbitrary vector $\underline{A}$ in terms of basis vectors $\hat{\underline{e}}_i$ which are orthogonal (and normalised);

$$\hat{\underline{e}}_m \cdot \hat{\underline{e}}_n = \delta_{m\,n} \,. \tag{266}$$

Have also seen that an arbitrary function of $\cos\theta$ can be expanded as a series of Legendre polynomials, which are orthogonal (and normalised);

$$\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = \frac{2}{2m+1}\, \delta_{m\,n} \,. \tag{267}$$

The two formulae look very similar. The crucial difference between the two is that the scalar product is defined differently in the two cases. For the Legendre polynomials, the scalar product of two of them is given by an integral from -1 to $+1$. The factor of $2/(2m+1)$ on the right hand side is of no real importance — one doesn't have to work with basis vectors of length one.

6.1 Fourier Series

The other place where you have met the expansion of functions in terms of orthogonal functions is in the Fourier series that you saw in the Waves and Optics course in the first year. A pure note on a violin corresponds to a sinusoidal variation in both position x and time t. However, when the violinist bows the instrument, she or he excites a whole range of notes. To find the notes, the initial signal must be expanded in a Fourier series. We want here to justify and extend some of these first year techniques.

The orthogonality integral is most elegant in terms of complex exponentials:

$$\int_{-\pi}^{+\pi} e^{-imx}\, e^{inx}\, dx = 2\pi\delta_{m\,n} \,. \tag{268}$$

Using Euler's relation, $e^{imx} = \cos mx + i\sin mx$, we can then convert Eq. (268) into integrals for sines and cosines to give

$$\int_{-\pi}^{+\pi} \cos mx\, \sin nx\, dx = 0 \quad \text{for all } m, n. \tag{269}$$

Provided that $m, n \neq 0$,

$$\int_{-\pi}^{+\pi} \cos mx\, \cos nx\, dx = \int_{-\pi}^{+\pi} \sin mx\, \sin nx\, dx = \pi\, \delta_{m\,n} \,. \tag{270}$$

If $m = n = 0$, the the sine integral vanishes and we are just left with the cosine integral

$$\int_{-\pi}^{+\pi} 1\, dx = 2\pi \; . \tag{271}$$

Thus the functions

$$\frac{1}{\sqrt{2\pi}} \; , \quad \frac{1}{\sqrt{\pi}} \cos mx \; , \quad \frac{1}{\sqrt{\pi}} \sin mx \; , \tag{272}$$

with m a positive integer, form an orthonormal set with respect to integration over the interval $-\pi \leq x \leq +\pi$.

An arbitrary function $f(x)$ in the interval $-\pi \leq x \leq +\pi$ may be written in the form

$$f(x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \, \cos nx + \sum_{n=1}^{\infty} b_n \, \sin nx \; , \tag{273}$$

where the Fourier coefficients are given by

$$\begin{aligned} a_m &= \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \, \cos mx \, dx \; , \\ b_m &= \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \, \sin mx \, dx \; . \end{aligned} \tag{274}$$

Proof: Multiply both sides of Eq. (273) by $\sin mx$ and integrate from $-\pi$ to $+\pi$.

$$\begin{aligned} \int_{-\pi}^{+\pi} f(x) \, \sin mx \, dx = \tfrac{1}{2} a_0 \int_{-\pi}^{+\pi} \sin mx \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{+\pi} \sin mx \, \cos nx \, dx \\ + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{+\pi} \sin mx \, \sin nx \, dx \; . \end{aligned} \tag{275}$$

The first two terms on the right hand side clearly vanish because the integrands are odd. The third term is only non-zero if $m = n$, which means that there is only one term in the sum. Using Eq. (270), this gives

$$\int_{-\pi}^{+\pi} f(x) \, \sin mx \, dx = \sum_{n=0}^{\infty} b_n \, \pi \, \delta_{m\,n} = \pi \, b_m \; , \tag{276}$$

as required.

Alternatively, multiplying by the cosine,

$$\begin{aligned} \int_{-\pi}^{+\pi} f(x) \, \cos mx \, dx = \tfrac{1}{2} a_0 \int_{-\pi}^{+\pi} \cos mx \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{+\pi} \cos mx \, \cos nx \, dx \\ + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{+\pi} \cos mx \, \sin nx \, dx \; . \end{aligned} \tag{277}$$

It is the third term which now vanishes for all m and n. If $m = 0$, only the first term survives and

$$\int_{-\pi}^{+\pi} f(x)\, \cos mx\, dx = \tfrac{1}{2} a_0\, 2\pi = \pi\, a_0\,, \quad \text{for } m = 0\,. \tag{278}$$

On the other hand, if $m \neq 0$, it is the second term which is non-vanishing and

$$\int_{-\pi}^{+\pi} f(x)\, \cos mx\, dx = \sum_{n=1}^{\infty} a_n\, \pi\, \delta_{m\,n} = \pi\, a_m\,. \tag{279}$$

The $\frac{1}{2}$ factor in front of a_0 in Eq. (274) gives a consistent formula for all a_m.

Example 1: Rectangular wave. Consider the function

$$f_1(x) = \begin{cases} +1 & \text{for } \; 0 < x < \pi \\ -1 & \text{for } \; -\pi < x < 0 \end{cases}$$

Note that this is an <u>odd</u> function, *i.e.* $f(x) = -f(-x)$. Using Eq. (274), this means that all the even coefficients $a_n = 0$. [The $\cos nx$ are even functions and, when multiplied by $f(x)$, give odd integrands.] On the other hand

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin nx\, dx - \frac{1}{\pi}\int_{-\pi}^{0} \sin nx\, dx = \frac{1}{n\pi}\Big[-\cos nx\Big]_0^{\pi} - \frac{1}{n\pi}\Big[-\cos nx\Big]_{-\pi}^{0}$$

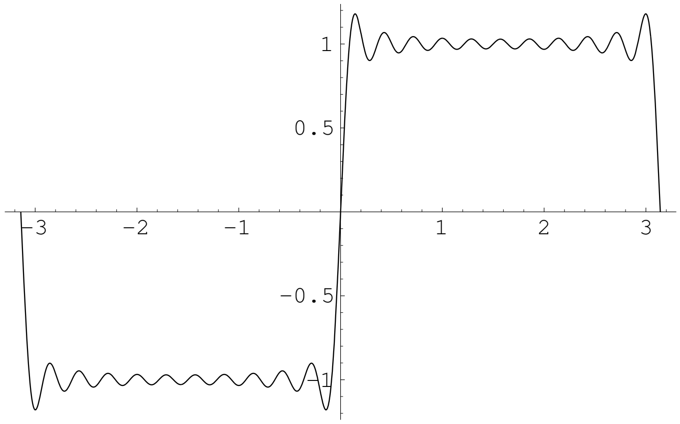
$$= \frac{1}{n\pi}[1 - \cos n\pi + 1 - \cos(-n\pi)] = \frac{2}{n\pi}[1 - \cos n\pi] = \frac{4}{n\pi} \times \begin{cases} 1 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

Thus the Fourier series becomes

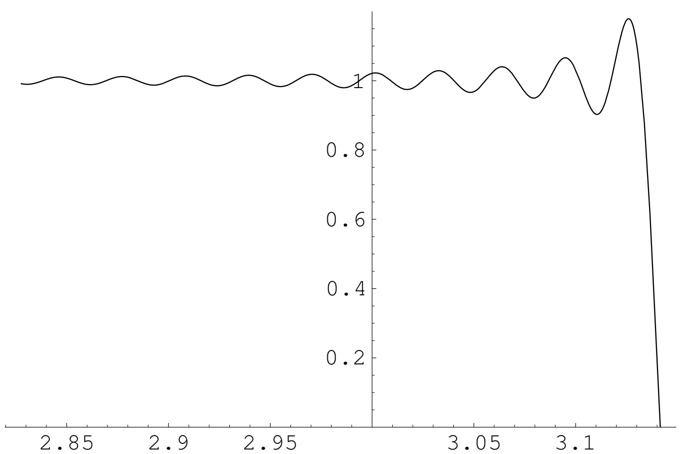
$$f_1(x) = \frac{4}{\pi} \sum_{n\ \mathrm{odd}} \frac{1}{n} \sin nx\,.$$

As an example of how things look if we just take a finite number of terms, the picture shows what happens when the series is truncated at $n = 21$. You see there are lots of little oscillations (typically 21) and the sizes of these oscillations get smaller as the number of terms increases. However, with a finite number of terms like this, the representation of a function which changes so sharply near $x = 0$ is not perfect! Note that the original function was not defined at $x = 0$ but the Fourier series has resulted in a representation with $f(0) = 0$. This is typical of a case where the function is discontinuous and the Fourier series will then converge to the mean of the results to the left and right of the discontinuity at $x = x_0$:

$$\longrightarrow \lim_{\epsilon \to 0} \tfrac{1}{2} \left\{ f(x_0 + \epsilon) + f(x_0 - \epsilon) \right\}\,. \tag{280}$$



No matter how many terms you add, you never get it quite right at a discontinuity. Thus in the next picture there are are 100 terms and there is still an overshoot of about 18%. With more terms, the overshoot always stays the same size but it just gets squeezed into a smaller region in x. This is known as the Gibbs phenomenon. On the other hand, one could not get anything like as good a description of a discontinuous function using a power series.



Periodic Functions

So far we have looked at functions defined in the region $-\pi \le x \le +\pi$. What happens outside this region? Going back to Eq. (273), we see that the function is periodic with period 2π since

$$f(x+2\pi) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n(x+2\pi) + \sum_{n=1}^{\infty} b_n \sin n(x+2\pi)$$

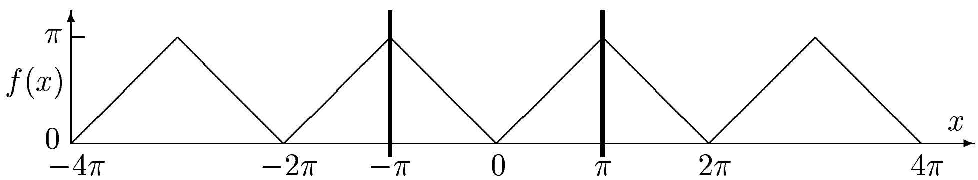
$$= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx = f(x)\,. \tag{281}$$

Therefore we can use Fourier series either on a function which is defined just over an interval $-\pi \le x \le +\pi$, or one which is periodic with period 2π.

Example 2: A function $f_2(x)$, which is periodic with period 2π, is defined by

$$\begin{aligned} f_2(x) &= x\,, & 0 \le x \le \pi\,, \\ &= -x\,, & -\pi \le x \le 0\,. \end{aligned}$$

$f_2(x)$ is an even function, as shown in the picture.



Due to the eveness, the b_n all vanish since the integrands are odd. This is a very general trick — an odd function only has sines in its expansion whereas an even function has only cosines. The even coefficients are:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \left[\frac{x^2}{\pi}\right]_0^{\pi} = \pi\,,$$

where use has been made of the even character to integrate over half the interval.

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi}\,.$$

The sine function vanishes at both limits, so that

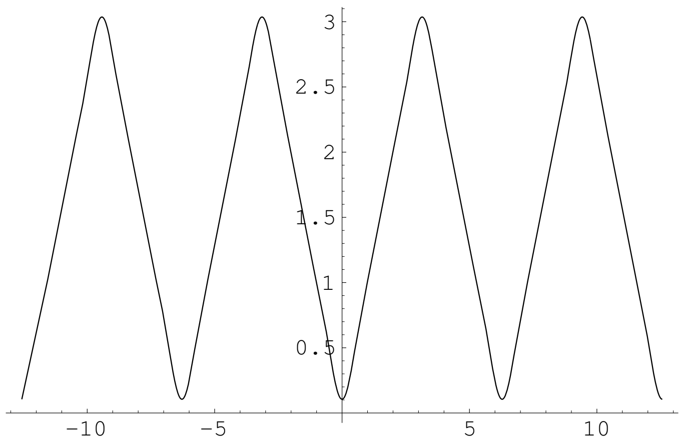
$$a_n = -\frac{2}{\pi n^2}\left[1-(-1)^n\right]\,.$$

This means that $a_n = -4/\pi n^2$ if n is odd but that $a_n = 0$ for even n.

Hence

$$f_2(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2} \cos nx \,.$$

This series converges rather faster than the first example since, keeping terms only up to $n = 5$, we get the picture:



The reason for this better behaviour is that, unlike the case in example 1, the original function $f(x)$ has no sudden jumps, although it has sudden changes in slope. Note, however, that with a finite number of terms the Fourier series never quite gets to zero.

Since $f_2(x)$ should vanish at $x = 0$, rearranging the Fourier series at this point gives

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8} \,.$$

The right hand side ≈ 1.234. Keeping only three terms on the left hand side gives $1 + 1/9 + 1/25 \approx 1.151$, which differs from the true answer by about 7%. This is another manifestation of the convergence of the Fourier series.

General interval

So far only looked at x between $\pm\pi$ but the same formulae are valid for any range of the same size, such as $0 \leq x \leq 2\pi$. If, on the other hand, the fundamental interval is $-L \leq x \leq +L$, just change variables to $y = \pi x/L$. Then

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \tfrac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi x}{L} \,, \tag{282}$$

where the Fourier coefficients are given by

$$a_m = \frac{1}{L}\int_{-L}^{+L} f(x)\,\cos\tfrac{m\pi x}{L}\,dx\,,$$
$$b_m = \frac{1}{L}\int_{-L}^{+L} f(x)\,\sin\tfrac{m\pi x}{L}\,dx\,. \qquad (283)$$

The conditions under which a Fourier expansion is valid go by the name of Dirichlet conditions, which are discussed by Boas. Roughly speaking, if $f(x)$ is periodic with period $2L$ with a finite number of discontinuities, then the expansion is valid provided that

$$\int_{-L}^{+L} \mid f(x) \mid \, dx \quad \text{is finite.} \qquad (284)$$

Differentiation of Fourier series

General motto: *be careful* ! In the examples that we have looked at,

$$f_1(x) = \frac{d}{dx}\,f_2(x)\,.$$

Does this hold for their Fourier series? Just try and see!

$$\frac{d}{dx}\,f_2(x) = \frac{d}{dx}\left(\frac{\pi}{2} - \frac{4}{\pi}\sum_{n\text{ odd}}\frac{1}{n^2}\cos nx\right) = \frac{4}{\pi}\sum_{n\text{ odd}}\frac{1}{n}\sin nx = f_1(x)\,,$$

which works fine.

Now go one step further and look at the Fourier series for the next derivative

$$\frac{d}{dx}\,f_1(x) = \frac{4}{\pi}\sum_{n\text{ odd}}\cos nx\,.$$

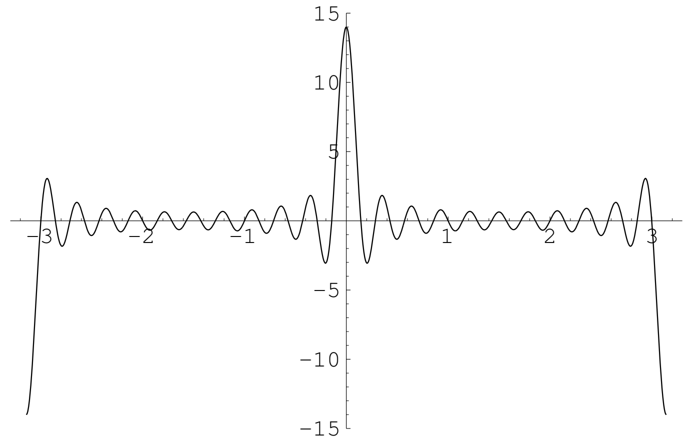
The series clearly must blow up at $x = 0$ or $x = \pm\pi$ because we then have an infinite number of terms equal to $+1$ or -1. Away from these points, the Fourier series oscillates around zero as in the picture, which was calculated with terms up to $n = 21$. If one takes more terms the peaks at $x = m\pi$ get higher but narrower.

To repeat, if the function is smooth then we can differentiate its Fourier series term by term. At any discontinuities we have to be careful — sometimes very careful!

Parseval's Identity

Suppose that $f(x)$ is periodic with period 2π such that it has the Fourier series

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\,\cos nx + \sum_{n=1}^{\infty} b_n\,\sin nx\,. \qquad (285)$$



Parseval's theorem is that the average value of f^2 is given by

$$< f^2(x) >= \frac{1}{2\pi} \int_{-\pi}^{+\pi} [f(x)]^2 \, dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \, . \tag{286}$$

We can insert the representation of Eq. (285) into the left hand side of Eq. (286) and simply carry out all the integrations. Now, by Eqs. (269, 270), there are no cross terms which survive the integration, so that

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} [f(x)]^2 \, dx = \frac{1}{2\pi} \int_{-\pi}^{+\pi} dx \left\{ \left(\tfrac{1}{2} a_0\right)^2 + \sum_{n=1}^{\infty} a_n^2 \cos^2 nx + \sum_{n=1}^{\infty} b_n^2 \sin^2 nx \right\} . \tag{287}$$

Now the average value $< \cos^2 nx >=< \sin^2 nx >= \frac{1}{2}$, so that Parseval's identity follows immediately.

Let us now see what this gives us for the two examples that we have worked out. In the first case, $f_1^2(x) = 1$, and hence

$$1 = \frac{1}{2} \frac{16}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \, ,$$

that is

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8} \, .$$

But this is already the result that we got from looking at the sum of the Fourier series in the second example at the position $x = 0$. The two answers are the same!

In the second case of the saw-tooth wave,

$$< f_2^2(x) >= \frac{1}{2\pi} \int_{-\pi}^{+\pi} x^2 \, dx = \left[\frac{x^3}{6\pi}\right]_{-\pi}^{+\pi} = \frac{\pi^2}{3} \, .$$

Hence

$$\frac{\pi^2}{3} = \left(\frac{\pi}{2}\right)^2 + \frac{8}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^4} \, .$$

Thus

$$\sum_{n \text{ odd}} \frac{1}{n^4} = \frac{\pi^4}{96} \, .$$

This agrees with the output from Mathematica!

Complex Fourier series

For those of you who are happy with complex numbers, the complex Fourier series are easier to handle than the real ones. If $f(x)$ is periodic, with period 2π, then

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n \, e^{inx} \, , \tag{288}$$

where the *complex* coefficients are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{-inx} \, f(x) \, dx \, . \tag{289}$$

The proof of this follows immediately using the orthonormality condition of Eq. (268):

$$\int_{-\pi}^{+\pi} e^{-imx} \, e^{inx} \, dx = 2\pi \delta_{mn} \, .$$

Example: Let us do example 1 again and show that we get the same answer.

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{0} e^{-inx} \, (-1) \, dx + \frac{1}{2\pi} \int_{0}^{+\pi} e^{-inx} \, (+1) \, dx = \frac{1}{2\pi} \left[\frac{-1}{-in} \, e^{-inx} \right]_{-\pi}^{0} + \frac{1}{2\pi} \left[\frac{1}{-in} \, e^{-inx} \right]_{0}^{\pi}$$

$$= \frac{1}{n\pi i} \left(1 - (-1)^n \right) \, .$$

This means that all the even coefficients vanish and the odd ones are $c_n = 2/(n\pi i)$. The complex Fourier series becomes

$$f(x) = \frac{2}{\pi i} \sum_{n=-\infty}^{+\infty} \frac{1}{n} \, e^{inx} \quad (n \text{ odd}) = \frac{2}{\pi i} \sum_{n=-\infty}^{-1} \frac{1}{n} \, e^{inx} + \frac{2}{\pi i} \sum_{n=1}^{+\infty} \frac{1}{n} \, e^{inx}$$

$$= -\frac{2}{\pi i} \sum_{n=1}^{\infty} \frac{1}{n} \, e^{-inx} + \frac{2}{\pi i} \sum_{n=1}^{+\infty} \frac{1}{n} \, e^{inx} = \frac{4}{\pi} \sum_{n=1}^{+\infty} \frac{1}{n} \, \frac{1}{2i} \left(e^{inx} - e^{-inx} \right) \, ,$$

where all the sums are over the odd values of n. You recognise the expression for $\sin nx$ in the last bracket and so we have obtained the same result as before.

There is a form of Parseval's identity which is valid for complex Fourier series:

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} \mid f(x) \mid^2 \, dx = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f^*(x) \, f(x) \, dx = \sum_{n=-\infty}^{+\infty} \mid c_n \mid^2 \, . \tag{290}$$

6.2 Fourier Transforms

Go back to the expressions of eqs. (282, 283) for the Fourier series and Fourier coefficients in the case of an arbitrary interval of length $2L$:

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\tfrac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\tfrac{n\pi x}{L} \,,$$

where

$$\begin{aligned} a_m &= \frac{1}{L}\int_{-L}^{+L} f(t)\,\cos\tfrac{n\pi t}{L}\,dt\,, \\ b_m &= \frac{1}{L}\int_{-L}^{+L} f(t)\,\sin\tfrac{n\pi t}{L}\,dt. \end{aligned}$$

Putting these together as one equation,

$$f(x) = \frac{1}{2L}\int_{-L}^{+L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{\infty}\cos\tfrac{n\pi x}{L}\int_{-L}^{+L} f(t)\,\cos\tfrac{n\pi t}{L}dt + \frac{1}{L}\sum_{n=1}^{\infty}\sin\tfrac{n\pi x}{L}\int_{-L}^{+L} f(t)\,\sin\tfrac{n\pi t}{L}dt. \tag{291}$$

Using the trigonometric addition formula

$$\cos\tfrac{n\pi x}{L}\cos\tfrac{n\pi t}{L} + \sin\tfrac{n\pi x}{L}\sin\tfrac{n\pi t}{L} = \cos\tfrac{n\pi}{L}(t-x)\,, \tag{292}$$

the Fourier series result can be written in the more compact form

$$f(x) = \frac{1}{2L}\int_{-L}^{+L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{+L} f(t)\,\cos\tfrac{n\pi}{L}(t-x)\,dt\,. \tag{293}$$

Fourier transforms are what happens to Fourier series when the interval length $2L$ tends to infinity. To bring this about, define

$$\omega = \frac{n\pi}{L} \quad\text{and}\quad \Delta\omega = \frac{\pi}{L}\,. \tag{294}$$

In the limit that $L \to \infty$, the first term in Eq. (293) goes to zero provided the infinite integral converges. Hence

$$f(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\Delta\omega\int_{-\infty}^{+\infty} f(t)\,\cos\omega(t-x)\,dt\,. \tag{295}$$

Now since $\Delta\omega \to 0$ as $L \to \infty$, the sum in Eq. (295) can be replaced by an integral to reveal the fundamental expression of Fourier transforms:

$$f(x) = \frac{1}{\pi}\int_0^{\infty} d\omega\int_{-\infty}^{+\infty} f(t)\,\cos\omega(t-x)\,dt\,. \tag{296}$$

Since $\cos\omega(t-x)$ is an *even* function of ω, we can extend the integration limit to $-\infty$ provided that we divide by a factor of 2:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(t)\,\cos\omega(t-x)\,dt\,. \tag{297}$$

On the other hand, $\sin\omega(t-x)$ is an *odd* function of ω, which means that

$$0 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(t)\,\sin\omega(t-x)\,dt\,. \tag{298}$$

Adding i times Eq. (298) to Eq. (297), we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(t)\,[\cos\omega(t-x)+i\sin\omega(t-x)]\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(t)\,e^{i\omega(t-x)}\,dt. \tag{299}$$

Splitting up the exponential, we get to the final result that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\,e^{-i\omega x} \int_{-\infty}^{+\infty} f(t)\,e^{i\omega t}\,dt\,. \tag{300}$$

Now introduce the Fourier transform

$$g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\,e^{i\omega x}\,dx\,, \tag{301}$$

and its inverse

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(\omega)\,e^{-i\omega x}\,d\omega\,. \tag{302}$$

The two equations look the same except that in one there is a $+i$ in the exponent, whereas in the other there is a $-i$. Eqs.(301, 302) have been defined symmetrically, each with a $1\sqrt{2\pi}$ factor. Many books put a $1/2\pi$ factor in front of one integral and unity in front of the other.

The variable ω introduced here is an arbitrary mathematical variable but in most physical problems it corresponds to the angular frequency ω. The Fourier transform represents $f(x)$ in terms of a distribution of infinitely long sinusoidal wave trains where the frequency is a continuous variable. You will come across this in Quantum Mechanics, where such waves are eigenfunctions of the momentum operator $\hat{p}$. Then $g(\omega) = g(p)$ is the momentum-space representation of the function $f(x)$.

Example

Consider

$$E(t) = E_0\,e^{-\gamma t/2}\,e^{-i\omega_0 t} = E_0\,e^{-(i\omega_0-\frac{1}{2}\gamma)t} \quad \text{for } t \geq 0$$

and which vanishes for negative values of t. This could represent a damped oscillating electric field which was switched on at time $t = 0$. The Fourier transform is

$$g(\omega) = \frac{1}{\sqrt{2\pi}} E_0 \int_0^{+\infty} e^{i\omega t}\, e^{-i(\omega_0 - \frac{1}{2}i\gamma)t}\, dt = \frac{1}{\sqrt{2\pi}} E_0 \frac{1}{i\omega - i\omega_0 + \frac{1}{2}\gamma} \left[e^{i(\omega - \omega_0 + \frac{1}{2}i\gamma)t} \right]_0^{\infty}$$

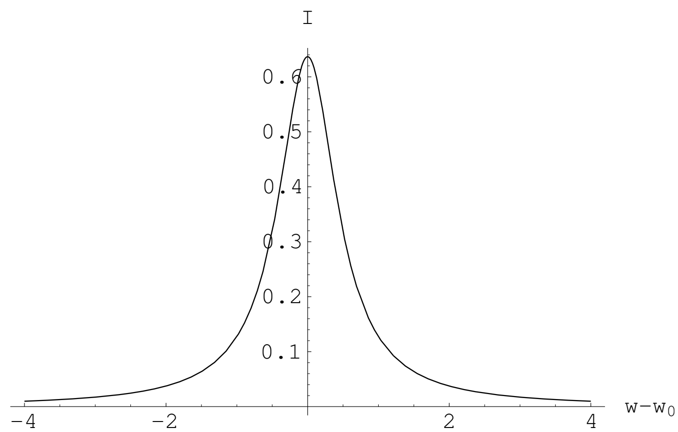
Because of the damping, the integated term vanishes at the upper limit and so we are left with

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \frac{iE_0}{\omega - \omega_0 + \frac{1}{2}i\gamma} \cdot$$

The intensity spectrum

$$I(\omega) = |\, g(\omega)\,|^2 = \frac{E_0^2}{2\pi} \frac{1}{(\omega - \omega_0)^2 + \gamma^2/4}$$

is peaked at $\omega = \omega_0$ with a width of γ. In plotting the figure we have taken $\gamma = 1$.



Useful results

1) If $f(x)$ is an *even* function of x, then

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, \cos\omega x\, dx$$

is an *even* function of ω.

2) Similarly, if $f(x)$ is an *odd* function of x, then $g(\omega)$ is an *odd* function of ω:

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{i\omega x}\, dx = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, \sin\omega x\, dx\,.$$

3) Differentiating Eq. (302),

$$f'(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} [-i\omega\, g(\omega)]\, e^{-i\omega x}\, d\omega\,, \qquad (303)$$

so that $-i\omega\, g(\omega)$ is the Fourier transform of $f'(x)$. By extension, $(-i\omega)^n\, g(\omega)$ is the Fourier transform of $d^n f/dx^n$.

4) From Eq. (302),

$$f(x+a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} [e^{-i\omega a}\, g(\omega)]\, e^{-i\omega x}\, d\omega\ , \tag{304}$$

so that $e^{-i\omega a}\, g(\omega)$ is the Fourier transform of $f(x+a)$.

The Dirac delta function

The Kronecker delta symbol $\delta_{i\,j}$ has the property that

$$a_i = \sum_i \delta_{i\,j}\, a_j \tag{305}$$

for any vector a_j, provided that the sum includes the term where $i = j$. The Dirac delta function is the generalisation of this to the case where we have an integral rather than a sum, *i.e.* we want a function $\delta(x-t)$, such that

$$f(x) = \int_{-\infty}^{+\infty} \delta(x-t)\, f(t)\, dt\,. \tag{306}$$

This means that $\delta(t-x)$ is zero everywhere except the point $t = x$ but there it is so big that the integral is unity. This is rather like having a point charge in electrostatics — it is an idealisation. $\delta(t-x)$ is not a function in the normal sense; it is just too badly behaved.

Going back to Eq. (300),

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega\, e^{-i\omega x} \int_{-\infty}^{+\infty} f(t)\, e^{i\omega t}\, dt\ ,$$

we can rearrange it as

$$f(x) = \int_{-\infty}^{+\infty} f(t)\, dt \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega\, e^{i\omega(t-x)} \right\}\ . \tag{307}$$

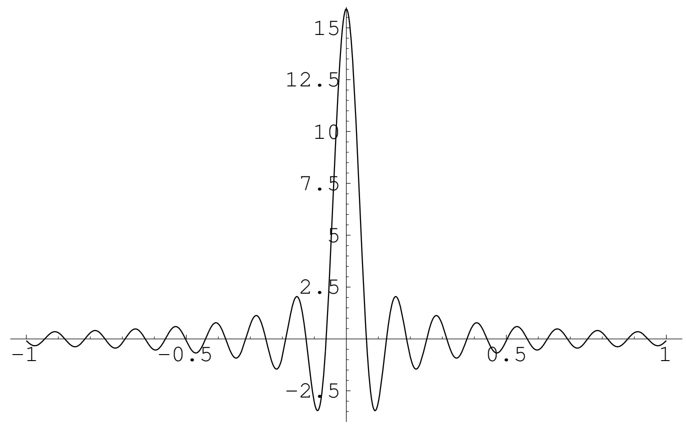
Thus we can identify

$$\delta(t-x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega\, e^{i\omega(t-x)}\ . \tag{308}$$

We have to be a bit careful about the convergence of the integral at large values of ω. Let us cut off the integration at $\omega = \pm N$ and then study what happens as N gets large.

$$\delta_N(t-x) = \frac{1}{2\pi} \int_{-N}^{+N} d\omega\, e^{i\omega(t-x)} = \frac{\sin N(t-x)}{\pi(t-x)}\ . \tag{309}$$

With $N = 50$, the figure shows a strong spike at $t = x$, but with lots of oscillations. The spike gets sharper as N gets larger, but the lobes at the bottom remain a constant fraction $2/3\pi$ of the central value. Note that $\delta(x) = \delta(-x)$; the Dirac delta is an even function.



Parseval's theorem

The equivalent of Parseval's theorem for Fourier transforms is easily proved using the Dirac delta-function. From Eq.(301),

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{i\omega x}\, dx \,,$$

$$g^*(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f^*(y)\, e^{-i\omega y}\, dy \,, \tag{310}$$

where x is replaced by y in the second integral for clarity. Multiply the two expressions together and integrate over ω.

$$\int_{-\infty}^{\infty} g^*(\omega)\, g(\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f^*(y)\, dy \int_{-\infty}^{+\infty} f(x)\, dx \int_{-\infty}^{+\infty} e^{i\omega(x-y)}\, d\omega \,. \tag{311}$$

But the last integral is just $2\pi\, \delta(y - x)$. The delta function removes the y integration and puts $y = x$ everywhere. The 2π factor knocks out the $1/2\pi$ factor outside and we are left with

$$\int_{-\infty}^{\infty} g^*(\omega)\, g(\omega)\, d\omega = \int_{-\infty}^{\infty} f^*(x)\, f(x)\, dx \,. \tag{312}$$

In words, the total intensity of a signal is equal to the total intensity of its Fourier transform. You should check this on the example given in class where

$$\mid E(t) \mid^2 = \mid E_0 \mid^2 \; e^{-\gamma t} \quad \text{and} \quad \mid g(\omega) \mid^2 = \frac{E_0^2}{2\pi} \frac{1}{(\omega - \omega_0)^2 + \gamma^2/4} \,.$$