5 Legendre Functions

Solve Legendre's differential equation

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-\mu^2}\right]\Theta = 0\,. \tag{196}$$

For $m = 0$ there is no azimuthal dependence on the angle ϕ:

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + \ell(\ell+1)\,\Theta = 0\,. \tag{197}$$

Special case of $\ell = 0$

Start with the even simpler case that we can treat by A-level methods. For $\ell = 0$,

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] = 0\,.$$

This means that the quantity inside the square bracket must be some constant C;

$$(1-\mu^2)\frac{d\Theta}{d\mu} = C\,.$$

This equation separates as

$$\int d\Theta = \int \frac{C}{(1-\mu^2)}\,d\mu\,,$$

giving the solution

$$\Theta = C\,\tfrac{1}{2}\ell n\left(\frac{1+\mu}{1-\mu}\right) + D\,. \tag{198}$$

Legendre equation is ordinary second-order DE. so solution contains two arbitrary integration constants, written here as C and D. There are two independent solutions of the equation

$$P_0(\mu) = 1\,, \tag{199}$$

$$Q_0(\mu) = \tfrac{1}{2}\ell n\left(\frac{1+\mu}{1-\mu}\right)\,, \tag{200}$$

where subscript denotes the value of ℓ.

Since Legendre equation is homogeneous, most general solution is a linear superposition of P_0 and Q_0,

$$\Theta(\mu) = C\,Q_0(\mu) + D\,P_0(\mu)\,.$$

Note that $Q_0(\mu)$ diverges at $\theta = 0$, *i.e.* $\mu = \cos\theta = +1$.

Away from $\ell = m = 0$, the solutions are more complicated. In general, one solutions is finite at $\mu = \pm 1$, whereas the other one blows up there. To find such solutions, we must apply series methods.

5.1 Series solution

First note the eq. remains unchanged if we let $\mu \to -\mu$. As before, this means can write independent solutions as either odd or even functions of μ. This condition is satisfied by solutions obtained for $\ell = m = 0$, (200).

Carrying out a differentiation, Legendre's eq. becomes

$$\frac{d^2\Theta}{d\mu^2} - \frac{2\mu}{(1-\mu^2)}\frac{d\Theta}{d\mu} + \frac{\ell(\ell+1)}{(1-\mu^2)}\Theta = 0\,, \tag{201}$$

which is ordinary at $\mu = 0$ but has regular singularities at $\mu = \pm 1$, expect series solutions about $\mu = 0$ will converge for $|\mu| < 1$.

Look for solutions in the series form

$$\begin{aligned} \Theta &= \sum_{n=0}^{\infty} a_n\,\mu^n\,, \\ \Theta' &= \sum_{n=0}^{\infty} n\,a_n\,\mu^{n-1}\,, \\ \Theta'' &= \sum_{n=0}^{\infty} n(n-1)\,a_n\,\mu^{n-2}\,. \end{aligned} \tag{202}$$

Inserting the expressions for Θ, Θ' and Θ'' into Eq. (201), we find

$$\sum_{n=0}^{\infty} n(n-1)\,a_n\left[\mu^{n-2} - \mu^n\right] - 2\sum_{n=0}^{\infty} n\,a_n\,\mu^n + \ell(\ell+1)\sum_{n=0}^{\infty} a_n\,\mu^n = 0\,. \tag{203}$$

Grouping together all similar powers of μ simplifies things a bit:

$$\sum_{n=0}^{\infty} n(n-1)\,a_n\,\mu^{n-2} = \sum_{n=0}^{\infty}\{n(n+1) - \ell(\ell+1)\}\,a_n\,\mu^n\,. \tag{204}$$

To get recurrence relation, change dummy variable $n \to n+2$ on left of Eq. (204):

$$\sum_{n=-2}^{\infty}(n+1)(n+2)\,a_{n+2}\,\mu^n = \sum_{n=0}^{\infty}\{n(n+1) - \ell(\ell+1)\}\,a_n\,\mu^n\,. \tag{205}$$

Comparing coefficients of powers of μ then gives

$$a_{n+2} = \frac{(n+1)n - \ell(\ell+1)}{(n+1)(n+2)}\,a_n\,. \tag{206}$$

Recurrence relation links terms differing by two units in n. As for harmonic oscillator equation, is a direct consequence of the DE being even under $\mu \to -\mu$, means that there are odd and even solutions of Legendre's equation.

Even solutions

$$a_{n+2} = \frac{n(n+1) - \ell(\ell+1)}{(n+1)(n+2)}\, a_n = \frac{(n-\ell)(n+\ell+1)}{(n+1)(n+2)}\, a_n \,. \tag{207}$$

Solution is therefore

$$p_\ell(\mu) = a_0 \left[1 - \frac{\ell(\ell+1)}{2!}\,\mu^2 + \frac{(\ell-2)(\ell)(\ell+1)(\ell+3)}{4!}\,\mu^4 + \cdots\right] . \tag{208}$$

Odd solutions

Starts from a_1 giving

$$q_\ell(\mu) = a_1 \left[\mu - \frac{(\ell-1)(\ell+2)}{3!}\,\mu^3 + \frac{(\ell-3)(\ell-1)(\ell+2)(\ell+4)}{5!}\,\mu^5 + \cdots\right] . \tag{209}$$

Clear that $p_\ell(\mu)$ is even function of μ, as a_1 etc $= 0$, whereas $q_\ell(\mu)$ is odd function. Most general solution is

$$f_\ell(\mu) = A\, p_\ell(\mu) + B\, q_\ell(\mu) \,. \tag{210}$$

Earlier found the explicit forms of the solutions for $\ell = 0$, *viz*

$$P_0(\mu) = 1 \qquad \text{and} \qquad Q_0 = \tfrac{1}{2}\ell n \left(\frac{1+\mu}{1-\mu}\right) .$$

Since $P_0(\mu)$ is even and $Q_0(\mu)$ is odd, want to identify P_0 with p_0 and Q_0 with q_0.

Putting $\ell = 0$ in Eq. (208),

$$p_0(\mu) = a_0 \left[1 - \frac{0(1)}{2!}\,\mu^2 + \frac{(-2)(0)(1)(3)}{4!}\,\mu^4 + \cdots\right] = a_0\, P_0(\mu) \,. \tag{211}$$

Every term (except first) has an ℓ factor which kills it.

Odd solution of Eq. (209) is a bit more complicated:

$$q_0(\mu) = a_0 \left[\mu - \frac{(-1)(2)}{3!}\,\mu^3 + \frac{(-3)(-1)(2)(4)}{5!}\,\mu^5 + \cdots\right] = a_0 \left[\mu + \frac{1}{3}\,\mu^3 + \frac{1}{5}\,\mu^5 + \cdots\right] , \tag{212}$$

which is the series expansion of $\frac{1}{2}\ell n \left(\frac{1+\mu}{1-\mu}\right)$.

Shows all is OK for $\ell = 0$. Get an infinite series for $q_0(\mu)$ but one which terminates for $p_0(\mu)$.

Does the infinite series converge? Apply D'Alembert ratio test to investigate.

5.2 Range of Convergence

Series goes up by steps of two in n. For D'Alembert ratio test have then to compare $(n+2)$'nd term with n'th

$$R = \left|\frac{a_{n+2}\,\mu^{n+2}}{a_n\,\mu^n}\right| = \left|\frac{(n+1)n - \ell(\ell+1)}{(n+1)(n+2)}\,\mu^2\right| . \tag{213}$$

Convergence depends upon large n, where

$$R \longrightarrow \left[1 - \frac{2}{n}\right] \mu^2 \longrightarrow \mu^2 \,. \tag{214}$$

$R < 1$ guarantees convergence and this is true if $|\mu| < 1$, exactly as expected because of the regular singular points of Legendre equation at $\mu = \pm 1$.

To see that $Q_0(\mu) = \frac{1}{2}\ell n\left(\frac{1+\mu}{1-\mu}\right)$ does indeed blow up at $\mu = \pm 1$; just put it into your calculator and see the error message flashing! Important point to note that $\mu = \cos\theta = \pm 1$ corresponds to $\theta = 0°$ and $180°$, need finite answers at these two points. Why should the electrostatic potential be infinite at the top and bottom of a sphere?

Avoided this problem with $P_0(\mu) = 1$ because series *terminates* with finite number of terms (for $\ell = 0$ just a single one). No issue with convergence. Only way out; to get finite answers at $\mu = \pm 1$ series must terminate. End up with a polynomial rather than an infinite series.

Going back to recurrence relation of Eq. (206),

$$a_{n+2} = \frac{(n+1)n - \ell(\ell+1)}{(n+1)(n+2)}\, a_n \,,$$

series terminates if numerator on right hand side vanishes for some value of n;

$$(n+1)n - \ell(\ell+1) = (n-\ell)(n+1+\ell) = 0 \,. \tag{215}$$

The convention that $Re\{\ell\} \geq -\frac{1}{2}$ means that we need

$$\ell = n \,. \tag{216}$$

For even solution need ℓ to be any positive <u>even</u> integer,

For odd solution need ℓ to be any positive <u>odd</u> integer.

2B22 Quantum Mechanics course shows that condition ℓ be an integer corresponds to the quantisation of orbital angular momentum in integral units of $\hbar$. Result obtained here is fundamental to this, Atomic and other branches of Physics.

For any (non-negative) integer N, $p_{2N}(\mu)$ and $q_{2N+1}(\mu)$ are polynomials in μ, but that $p_{2N+1}(\mu)$ and $q_{2N}(\mu)$ are infinite series which diverge at $\mu = 1$. Clearly interested in solutions which are finite at $\theta = 0°$; group them with a common notation. Let

$$P_\ell(\mu) = \begin{cases} p_\ell(\mu) & \ell \text{ even}\,, \\ q_\ell(\mu) & \ell \text{ odd}\,, \end{cases}$$
$$Q_\ell(\mu) = \begin{cases} p_\ell(\mu) & \ell \text{ odd}\,, \\ q_\ell(\mu) & \ell \text{ even}\,. \end{cases} \tag{217}$$

$P_\ell(\mu)$ is a polynomial but $Q_\ell(\mu)$ is an infinite series which blows up at $\mu = 1$. Furthermore

$$\begin{aligned} P_\ell(-\mu) &= (-1)^\ell \, P_\ell(\mu) \, , \\ Q_\ell(-\mu) &= (-1)^{\ell+1} \, Q_\ell(\mu) \, . \end{aligned} \tag{218}$$

Summary

Only for non-negative integers ℓ do we have solutions of Legendre's equation which are finite at $\mu = \pm 1$. These are the Legendre polynomials $P_\ell(\mu)$. There are also Legendre functions of the second kind, $Q_\ell(\mu)$, but these blow up at $\mu = \pm 1$. The Q_ℓ are far less important in Physics and will be largely neglected.

Standard to "normalise" Legendre polynomials such that

$$P_\ell(1) = 1 \, . \tag{219}$$

From the series representation of Eqs. (208) and (209), we then see that

$$\begin{aligned} P_0(\mu) &= 1 \, , \\ P_1(\mu) &= \mu \, , \\ P_2(\mu) &= \tfrac{1}{2}(3\mu^2 - 1) \, . \end{aligned} \tag{220}$$

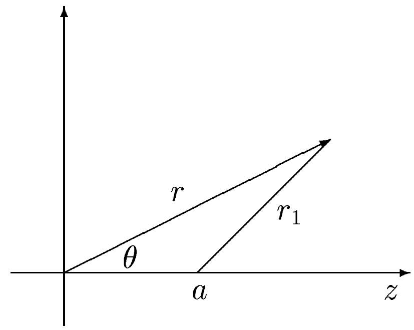
Going back to Eq. (124), original Laplace equation in spherical coordinates, most general solution which has <u>no</u> ϕ dependence is

$$V(r\,,\,\theta) = \sum_{\ell=0}^{\infty} \left[\alpha_\ell \, r^\ell + \beta_\ell \, r^{-\ell-1}\right] \; P_\ell(\cos\theta) \, , \tag{221}$$

where sum is over discrete integers ℓ, and α_ℓ and β_ℓ are constants fixed by boundary conditions.

5.3 Generating Function for Legendre Polynomials

Working out electrostatic potential due to point charge q at $z = a$.



In terms of the distance from the point $z = a$, potential is simply

$$V = \frac{q}{4\pi\varepsilon_0} \frac{1}{r_1} \, . \tag{222}$$

To evaluate it in terms of r and θ, use the cosine rule to obtain

$$r_1^2 = r^2 + a^2 - 2r\, a \cos\theta \, , \tag{223}$$

which leads to a potential of

$$V(r \, , \, \theta) = \frac{q}{4\pi\varepsilon_0} \left[r^2 + a^2 - 2r\, a\cos\theta\right]^{-\frac{1}{2}} \, . \tag{224}$$

There is no ϕ dependence because the charge was placed on the z-axis.

If we are interested in the potential in the region $r > a$, then we can expand Eq. (224) in powers of a/r,

$$\begin{aligned} V(r \, , \, \theta) &= \frac{q}{4\pi\varepsilon_0 \, r} \left[1 + \left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos\theta\right]^{-\frac{1}{2}} \, . \\ &\approx \frac{q}{4\pi\varepsilon_0 \, r} \left[1 - \frac{a^2}{2r^2} + \frac{a}{r}\cos\theta + \frac{3a^2}{2r^2}\cos^2\theta + \cdots\right] \\ &= \frac{q}{4\pi\varepsilon_0 \, r} \left[1 + \frac{a}{r}\cos\theta + \frac{a^2}{r^2}\,\tfrac{1}{2}\left(3\cos^2\theta - 1\right) + \cdots\right] \end{aligned} \tag{225}$$

Already know the general solution for Laplace's equation in any region where there is no charge. If the potential is to remain finite at large r, all the α_ℓ coefficients in Eq. (221) must vanish and so

$$V(r \, , \, \theta) = \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{\beta_\ell}{r^\ell} \, P_\ell(\cos\theta) \, . \tag{226}$$

To determine the values of the β_ℓ, look what happens at $\theta = 0$ where, by definition, $P_\ell(\cos\theta) = 1$. In the forward direction $r = r_1 + a$, and so

$$V(r\,,\,\theta) = \frac{1}{r}\sum_{\ell=0}^{\infty}\frac{\beta_\ell}{r^\ell} = \frac{q}{4\pi\varepsilon_0\,r}\,\frac{1}{(1-a/r)} = \frac{q}{4\pi\varepsilon_0\,r}\sum_{\ell=0}^{\infty}\frac{a^\ell}{r^\ell}\,. \tag{227}$$

Comparing different powers of r in the two sums, can read off

$$\beta_\ell = \frac{q}{4\pi\varepsilon_0}\,a^\ell\,. \tag{228}$$

Final solution at all angles

$$V(r\,,\,\theta) = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{\sqrt{r^2+a^2-2r\,a\cos\theta}} = \frac{q}{4\pi\varepsilon_0\,r}\sum_{\ell=0}^{\infty}\frac{a^\ell}{r^\ell}\,P_\ell(\cos\theta)\,. \tag{229}$$

Have solved a problem in electrostatics but result gives general method to derive Legendre polynomials. Comparing the two expressions for the potential in Eq. (229), and dropping elecrostatics factor outside gives

$$\frac{1}{r}\,\frac{1}{\sqrt{1+a^2/r^2-2\,(a/r)\,\cos\theta}} = \frac{1}{r}\sum_{\ell=0}^{\infty}\frac{a^\ell}{r^\ell}\,P_\ell(\cos\theta)\,. \tag{230}$$

Change to notation where $t = a/r$ and $x = \cos\theta$,

$$g(x,t) \equiv \frac{1}{\sqrt{1-2x\,t+t^2}} = \sum_{\ell=0}^{\infty} P_\ell(x)\,t^\ell\,. \tag{231}$$

This is the generating function for the Legendre polynomials. Only valid if $\mid t \mid < 1$, which corresponds to $r > a$, otherwise there are convergence problems.

If you expand the square root using the binomial expansion, and compare powers of t, then you get the same answers as we got before, *viz*

$$\begin{aligned} P_0(x) &= 1\,,\\ P_1(x) &= x\,,\\ P_2(x) &= \tfrac{1}{2}(3x^2-1)\,. \end{aligned} \tag{232}$$

etc.

5.4 Recurrence Relations

Apart from physical interpretation, generating function helps derive recurrence relations between Legendre polynomials. In practice most efficient way of deriving polynomials.

Differentiate the generating function of Eq. (231) partially with respect to t

$$\frac{\partial g(x,t)}{\partial t} = \frac{x-t}{(1-2x\,t+t^2)^{3/2}} = \sum_{n=0}^{\infty} n\,P_n(x)\,t^{n-1}\,. \tag{233}$$

Multiply both sides by $1-2x\,t+t^2$ to give

$$(x-t)\,\frac{1}{(1-2x\,t+t^2)^{\frac{1}{2}}} = (1-2x\,t+t^2)\sum_{n=0}^{\infty} n\,P_n(x)\,t^{n-1}\,. \tag{234}$$

On the left-hand side see once generating function for Legendre polynomials, which means

$$(x-t)\sum_{n=0}^{\infty} P_n(x)\,t^n = (1-2x\,t+t^2)\sum_{n=0}^{\infty} n\,P_n(x)\,t^{n-1}\,. \tag{235}$$

This equation is a power series in t which is supposed to be valid for a range of values of t. Consequently it must be valid separately for each power of t. Exactly same argument used in the series solution of DEs. Writing out explicitly all different powers,

$$x\sum_{n=0}^{\infty} P_n(x)\,t^n - \sum_{m=0}^{\infty} P_m(x)\,t^{m+1} = \sum_{\ell=0}^{\infty} \ell\,P_\ell(x)\,t^{\ell-1} - 2x\sum_{n=0}^{\infty} n\,P_n(x)\,t^n + \sum_{m=0}^{\infty} m\,P_m(x)\,t^{m+1}. \tag{236}$$

Formula written with different dummy indices ℓ, m, and n so can change a couple of them easily. Let

$$m \longrightarrow n-1\,,$$
$$\ell \longrightarrow n+1\,.$$

Then all terms in the sums contain same t^n factor. Reading off coefficient get

$$x\,P_n(x) - P_{n-1}(x) = (n+1)\,P_{n+1}(x) - 2n\,x\,P_n(x) + (n-1)\,P_{n-1}(x)\,.$$

Grouping like terms together gives the recurrence relation

$$(2n+1)\,x\,P_n(x) = (n+1)\,P_{n+1}(x) + n\,P_{n-1}(x)\,. \tag{237}$$

Thus, if you know $P_n(x)$ and $P_{n-1}(x)$, the recurrence relation allows you to obtain the formula for $P_{n+1}(x)$.

For example, putting $n=1$ in Eq. (237), we get

$$3x\,P_1(x) = 2P_2(x) + P_0(x)\,. \tag{238}$$

Since $P_0(x)=1$ and $P_1(x)=x$, this then immediately gives $P_2(x)=\frac{1}{2}\,(3x^2-1)$.

Using instead $n = 2$, we obtain

$$5x\, P_2(x) = 3P_3(x) + 2P_1(x)\,, \tag{239}$$

which means that

$$P_3(x) = \tfrac{1}{2}\,(5x^3 - 3x)\,.$$

5.5 Orthogonality of Legendre Polynomials

The Legendre differential equations for $P_n(x)$ and $P_m(x)$ are

$$\frac{d}{dx}\left[(1-x^2)\,P_n'(x)\right] + n(n+1)\,P_n(x) = 0\,,$$

$$\frac{d}{dx}\left[(1-x^2)\,P_m'(x)\right] + m(m+1)\,P_m(x) = 0\,. \tag{240}$$

Multiply the first of Eqs. (240) by $P_m(x)$ and the second by $P_n(x)$ and subtract one from the other to find:

$$P_m(x)\,\frac{d}{dx}\left[(1-x^2)\,P_n'(x)\right] - P_n(x)\,\frac{d}{dx}\left[(1-x^2)\,P_m'(x)\right] = [m(m+1) - n(n+1)]\; P_m(x)P_n(x).$$

Now integrate both sides of this expression over x from -1 to $+1$:

$$\int_{-1}^{+1}\left\{P_m(x)\,\frac{d}{dx}\left[(1-x^2)\,P_n'(x)\right] - P_n(x)\,\frac{d}{dx}\left[(1-x^2)\,P_m'(x)\right]\right\}\,dx$$

$$= [m(m+1) - n(n+1)]\int_{-1}^{+1} P_m(x)P_n(x)\,dx\,. \tag{241}$$

What we have to do now is show that the left hand side of Eq. (241) vanishes. This we do through integrating by parts.

$$\int_{-1}^{+1} P_m(x)\,\frac{d}{dx}\left[(1-x^2)\,P_n'(x)\right]\,dx = \left[P_m(x)\,(1-x^2)\,P_n'(x)\right]_{-1}^{+1} - \int_{-1}^{+1}(1-x^2)\,P_n'(x)P_m'(x)\,dx. \tag{242}$$

Now the first term on the RHS of Eq. (242) equals zero because $(1 - x^2) = 0$ at both the limits. On the other hand, the second term is cancelled by an identical one coming from the second term in Eq.(241) where $m \Leftrightarrow n$. Hence

$$[m(m+1) - n(n+1)]\int_{-1}^{+1} P_m(x)P_n(x)\,dx = 0\,, \tag{243}$$

which means that, if $n \neq m$,

$$\int_{-1}^{+1} P_m(x)P_n(x)\,dx = 0\,. \tag{244}$$

This the *Orthogonality relation.* It is analogous to orthogonality of two vectors except that have integral over a continuous variable rather than sum over components.

To construct equivalent of a unit vector, evaluate integral when $m = n$:

$$I_n = \int_{-1}^{+1} \left[P_n(x)\right]^2 \, dx \, . \tag{245}$$

Many ways of working this out, but easiest uses generating function of Eq. (231). Writing this twice gives

$$\frac{1}{\sqrt{1-2x\,t+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n \, ,$$

$$\frac{1}{\sqrt{1-2x\,t+t^2}} = \sum_{n=0}^{\infty} P_m(x)\, t^m \, . \tag{246}$$

Multiply these two expressions together to give a double summation over n and m.

$$\frac{1}{1-2x\,t+t^2} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x)\, P_m(x)\, t^{n+m} \, . \tag{247}$$

Now integrate over x from -1 to $+1$:

$$\int_{-1}^{+1} \frac{dx}{1-2x\,t+t^2} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \int_{-1}^{+1} P_n(x)\, P_m(x)\, t^{n+m} dx \, . \tag{248}$$

The integral on the left gives

$$\frac{1}{2t}\,\ell n\left(\frac{(1+t)^2}{(1-t)^2}\right) = \frac{1}{t}\,\ell n\left(\frac{1+t}{1-t}\right) = 2\sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1} \, . \tag{249}$$

Have already shown that integral on RHS vanishes unless $n = m$ and so only have a single sum:

$$\text{RHS} = \sum_{n=0}^{\infty} I_n\, t^{2n} \, . \tag{250}$$

Comparing coefficients of t^{2n} on left and right hand sides gives

$$I_n = \int_{-1}^{+1} \left[P_n(x)\right]^2 \, dx = \frac{2}{2n+1} \, . \tag{251}$$

The orthogonality and normalisation of Legendre polynomials can be written as

$$\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = \frac{2}{2m+1}\, \delta_{m\,n} \, , \tag{252}$$

where the Kronecker delta symbol for two integers m and n is defined by

$$\delta_{m\,n} = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \, . \end{cases} \tag{253}$$

5.6 Expansion in series of Legendre polynomials

Last year learned how to expand a function $f(x)$ in a Maclaurin series:

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \, x^n \,.$$

Also, in the first year Waves and Optics course, learned about expanding functions in series of sine and cosine functions. Such Fourier expansions will be the topic of the next section. Here expand $f(x)$ in an infinite series of Legendre polynomials:

$$f(x) = \sum_{n=0}^{\infty} a_n \, P_n(x) \,. \tag{254}$$

Start with a simple example:

$$f(x) = \frac{15}{2}x^2 - \frac{3}{2} = \frac{15}{2}\cdot\frac{1}{3}\left(2P_2(x) + P_0(x)\right) - \frac{3}{2}\,P_0(x) = P_0(x) + 5P_2(x)\,.$$

Whenever the power series for $f(x)$ only has a finite number of terms, *i.e.* is a polynomial, can calculate coefficients by solving system of algebraic equations. Example above is of this kind. If $f(x)$ is not a polynomial then can still calculate the coefficients using the orthonormality integral of Eq. (253). Multiplying Eq. (254) by $P_m(x)$ and integrating from -1 to $+1$ gives

$$\int_{-1}^{+1} f(x)\,P_m(x)\,dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{+1} P_n(x)\,P_m(x)\,dx = \sum_{n=0}^{\infty} a_n \,\frac{2}{2n+1}\,\delta_{m\,n} = \frac{2}{2m+1}\,a_m\,. \tag{255}$$

Hence

$$a_m = \frac{2m+1}{2}\int_{-1}^{+1} f(x)\,P_m(x)\,dx\,. \tag{256}$$

Example 1. Calculate the Legendre coefficients for $f(x) = \frac{15}{2}x^2 - \frac{3}{2}$. Putting in the explicit forms for the Legendre polynomials, we have

$$a_0 = \frac{1}{2}\int_{-1}^{+1}\left(\frac{15}{2}x^2 - \frac{3}{2}\right)\,dx = \frac{1}{2}\left(\frac{15}{3} - 3\right) = 1\,,$$

$$a_1 = \frac{3}{2}\int_{-1}^{+1}\left(\frac{15}{2}x^2 - \frac{3}{2}\right)\,x\,dx = 0 \quad \text{(integrand odd)},$$

$$a_2 = \frac{5}{2}\int_{-1}^{+1}\left(\frac{15}{2}x^2 - \frac{3}{2}\right)\left(\frac{3x^2}{2} - \frac{1}{2}\right)\,dx = \frac{5}{4}\left(\frac{45}{5} - \frac{24}{3} + 3\right) = 5\,.$$

These agree with what we found using direct algebra.

Example 2. Obtain the first two the Legendre coefficients for $f(x) = e^{\alpha x}$.

$$a_0 = \frac{1}{2}\int_{-1}^{+1} e^{\alpha x}\,dx = \frac{1}{2\alpha}\left(e^{\alpha} - e^{-\alpha}\right) = \frac{\sinh\alpha}{\alpha}\,,$$

$$a_1 = \frac{3}{2}\int_{-1}^{+1} x\,e^{\alpha x}\,dx = 3\left(\frac{\cosh\alpha}{\alpha} - \frac{\sinh\alpha}{\alpha^2}\right)\,,$$

where we had to do some integration by parts.

5.7 Return to the Potential Problem

Laplace equation in spherical polar coordinates and with axial symmetry (no ϕ dependence) has general solution for electrostatic potential in charge-free space

$$V(r, \cos\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)\,. \tag{257}$$

Suppose that $V(r) \to 0$ as $r \to \infty$. This means that all the $A_\ell = 0$ and

$$V(r, \cos\theta) = \sum_{\ell=0}^{\infty}\frac{B_\ell}{r^{\ell+1}}\,P_\ell(\cos\theta)\,. \tag{258}$$

To fix the values of B_ℓ coefficients, need another boundary condition:

Example

Suppose that on an isolated sphere of radius a the electrostatic potential varies like $V(r = a, \theta) = V_0\,e^{\alpha\cos\theta}$. How does the potential behave for large distances?

Using the Legendre series example already worked out,

$$B_0 = V_0 a\,\frac{\sinh\alpha}{\alpha}\,,$$

$$B_1 = 3V_0 a^2\left(\frac{\cosh\alpha}{\alpha} - \frac{\sinh\alpha}{\alpha^2}\right)\,,$$

and

$$V(r, \cos\theta) = V_0\left[\frac{a}{r}\,\frac{\sinh\alpha}{\alpha} + 3\frac{a^2}{r^2}\left(\frac{\cosh\alpha}{\alpha} - \frac{\sinh\alpha}{\alpha^2}\right)\cos\theta + 0\left(\frac{1}{r^3}\right)\right]\,.$$

For those of you familiar with electrostatics, the $\cos\theta$ term corresponds to the electric dipole moment and the discarded next term the quadrupole moment *etc.*

5.8 Associated Legendre Functions

In general ϕ dependence of solutions to Laplace's equation is of form $e^{im\phi}$, where m is an integer. To get the θ dependence, have to solve

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell^m}{dx}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]P_\ell^m=0\,. \tag{259}$$

Have here replaced $\mu\to x$ and called $\Theta(\mu)\to P_\ell^m(x)$.] Only for $m=0$ do we get the Legendre polynomials $P_\ell(x)$. To solve the equation for $m\neq 0$ is even more tedious than for $m=0$. But results are important for Quantum Mechanics, where ℓ is known as the angular momentum quantum number and m the magnetic quantum number.

Well behaved solutions of Legendre's equation are only possible if

- ℓ is a non-negative integer.
- m is an integer with $-\ell\le m\le\ell$.

Solutions for m and $-m$ are the same since only m^2 occurs in Legendre's equation. For $m>0$ the associated Legendre functions can be derived from the Legendre polynomials using

$$P_\ell^m(x)=(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^m P_\ell(x)\,. \tag{260}$$

The orthogonality relation is also a bit more complicated than that of Eq. (252):

$$\int_{-1}^{+1}P_\ell^m(x)\,P_n^m(x)\,dx=\frac{2}{2\ell+1}\,\frac{(\ell+m)!}{(\ell-m)!}\,\delta_{\ell n}\,. \tag{261}$$

Specific cases

$\ell=1,\ m=1$:

$$P_1^1(x)=(1-x^2)^{1/2}\frac{d}{dx}\,x=(1-x^2)^{1/2}=\sin\theta\,.$$

$\ell=2,\ m=1$:

$$P_2^1(x)=(1-x^2)^{1/2}\frac{d}{dx}\,\tfrac{1}{2}(3x^2-1)=3x\,(1-x^2)^{1/2}=3\sin\theta\cos\theta\,.$$

$\ell=2,\ m=2$:

$$P_2^2(x)=(1-x^2)\,\frac{d^2}{dx^2}\,\tfrac{1}{2}(3x^2-1)=3(1-x^2)=3\sin^2\theta\,.$$

As a couple of examples to check the orthogonality relations, consider

$$\int_{-1}^{+1}P_2^1(x)\,P_1^1(x)\,dx=\int_{-1}^{+1}3x(1-x^2)\,dx=0\,.$$

$$\int_{-1}^{+1}\left[P_2^1(x)\right]^2\,dx=\int_{-1}^{+1}9x^2(1-x^2)\,dx=2\,\frac{9}{3}-2\,\frac{9}{5}=\frac{12}{5}\,.$$

The last one agrees with the $\frac{2}{5}\,3!$ of Eq. (261).

5.9 Spherical Harmonics

In Quantum Mechanics, one often gets the θ and ϕ dependence packaged together as one function called a spherical harmonic $Y_\ell^m(\theta,\phi)$. Thus

$$Y_\ell^m(\theta,\phi) = c_{\ell,m}\, P_\ell^m(\cos\theta)\, e^{im\phi} \tag{262}$$

is a solution of Legendre equation. Here the constants $c_{\ell,m}$ could be chosen many ways.By convention Y_ℓ^m os normalised so that

$$\int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\phi\, Y_\ell^{m\,*}(\theta,\phi)\, Y_{\ell'}^{m'}(\theta,\phi) = \delta_{\ell,\ell'}\,\delta_{m,m'}\,. \tag{263}$$

Using Eq. (261), this is achieved with

$$c_{\ell,m} = (-1)^m \sqrt{\frac{(\ell-m)!\,(2\ell+1)}{(\ell+m)!\,4\pi}}\,. \tag{264}$$

In Quantum Mechanics you will at some stage need to remember the orthogonality/normalisation relation of Eq. (263) but you will <u>NOT</u> be required to remember the actual algebraic form of Eq. (264).

The first few spherical harmonics are

$$\begin{aligned}
Y_0^0(\theta,\phi) &= \frac{1}{\sqrt{4\pi}}\,,\\
Y_1^1(\theta,\phi) &= -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{i\phi}\,,\\
Y_1^0(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\,\cos\theta\,,\\
Y_1^{-1}(\theta,\phi) &= \sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{-i\phi}\,.
\end{aligned} \tag{265}$$