4 Series Solution of Differential Equations: Frobenius's Method

Have shown how and when differential equations (DEs) can be separated but need a general strategy for solving the resulting DEs.

Series solutions

Want to solve general, linear, homogenous, ordinary, second-order DE:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0\,. \tag{133}$$

One general method is expand y as a series:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{134}$$

about some point, x_0. For this method we need to:

1. What point, x_0, to use for the expansion;
2. Determine what values of a_n satisfy the DE;
3. Decide for what, if any, values of x the series converges;

4.1 Simple Series Solution of Second Order Equations

Use classical and quantal simple harmonic oscillator (HO) as an example.

Classical HO

Particle mass m; restoring force constant K; equation

$$m\frac{d^2y}{dx^2} = -Ky \tag{135}$$

or

$$\frac{d^2y}{dx^2} + \omega^2 x = 0; \qquad \omega = \left(\frac{K}{m}\right)^{\frac{1}{2}} \tag{136}$$

Most general solution is

$$y = A\,\cos\omega x + B\,\sin\omega x\,, \tag{137}$$

where A and B are arbitrary constants fixed by boundary conditions. A second order linear equation has two arbitrary constants.

The two individual solutions of this equation, *viz* $\cos\omega x$ and $\sin\omega x$, are respectively even and odd functions of the independent variable x. Why? Write Eq. (136) for a function $y = f(x)$ and then let $x \to -x$. Have the two equations

$$\begin{aligned} \frac{d^2 f(x)}{dx^2} + \omega^2 f(x) &= 0\,, \\ \frac{d^2 f(-x)}{dx^2} + \omega^2 f(-x) &= 0\,. \end{aligned} \tag{138}$$

Thus $f(-x)$ satisfies the same equation as $f(x)$ because all operators in Eq. (136) are even; $\dfrac{d^2}{dx^2}$ doesn't change when $x \to -x$. Any linear combinations of $f(x)$ and $f(-x)$ also satisfy the equations. In particular, the even and odd combinations

$$f_e(x) = \tfrac{1}{2}[f(x) + f(-x)]\,, \tag{139}$$

$$f_o(x) = \tfrac{1}{2}[f(x) - f(-x)] \tag{140}$$

also satisfy the equation. This is the real reason why $\cos\omega x$ and $\sin\omega x$ are solutions to the oscillator equation. This argument doesn't show that the basic solutions <u>have</u> to be either even or odd, but one can always choose them so to be. Will use this argument when we other DEs.

Now try for a series solution of the HO equation;

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n\, x^n\,, \\ \frac{dy}{dx} &= \sum_{n=0}^{\infty} (n)\, a_n\, x^{n-1}\,, \\ \frac{d^2y}{dx^2} &= \sum_{n=0}^{\infty} (n)(n-1)\, a_n\, x^{n-2}\,. \end{aligned} \tag{141}$$

Inserting these into Eq. (136), we find that

$$\sum_{n=0}^{\infty} (n)(n-1)\, a_n\, x^{n-2} + \omega^2 \sum_{n=0}^{\infty} a_n\, x^n = 0\,. \tag{142}$$

Changing the dummy index in the first sum by $n \to n+2$,

$$\sum_{n=-2}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n + \omega^2 \sum_{n=0}^{\infty} a_n\, x^n = 0\,, \tag{143}$$

all the powers now look the same; compare coefficients to obtain the recurrence relation

$$(n+2)(n+1)\, a_{n+2} + \omega^2\, a_n = 0\,, \tag{144}$$

$$a_{n+2} = -\frac{\omega^2}{(n+2)(n+1)}\, a_n\ . \tag{145}$$

Given the value of a_0, this allows us to evaluate a_2, and then a_4 *etc.* The odd a_n are completely independent and, as far as getting a solution is concerned, we can put them all to zero. This independence of the odd and even a_n is a consequence of the fact that odd and even solutions of the differential equation are possible. It therefore follows from the fact that the differential operator is even in x, as shown by Eq. (138). In order to generate these purely odd/even solutions, it is easiest to put $a_1 = 0$. Does not create extra solutions, merely mixes some of the odd solution with even ones.

The recurrence relation

$$a_{n+2} = -\frac{\omega^2}{(n+2)(n+1)}\, a_n\ . \tag{146}$$

has the solution for n even

$$\begin{aligned} a_n = &\ (-\omega^2)^{n/2}\, a_0/n! \qquad (n \text{ even})\ , \\ = &\qquad\quad 0 \qquad\qquad (n \text{ odd})\ . \end{aligned} \tag{147}$$

The total solution is then

$$y = a_0 \sum_{n\ even} (-1)^{n/2}(\omega x)^n\, \frac{1}{n!} = a_0\, \cos\omega x\ . \tag{148}$$

To generate the odd solutions, set $a_0 = 0$ and start from a_1. To do this set $b_m = a_{n+1}$ and $m = n - 1$.

The recurrence relation is

$$b_{m+2} = -\frac{\omega^2}{(m+3)(m+2)}\, b_m\ , \tag{149}$$

so that

$$\begin{aligned} b_m = &\ (-\omega^2)^{m/2}\, b_0/(m+1)! \qquad (m \text{ even})\ , \\ = &\qquad\quad 0 \qquad\qquad (m \text{ odd})\ , \end{aligned} \tag{150}$$

and

$$y = b_0\, x \sum_{m\ even} (-1)^{m/2}(\omega x)^m\, \frac{1}{(m+1)!} = \frac{b_0}{\omega}\, \sin\omega x\ . \tag{151}$$

Quantum HO

Potential corresponding to force $-Kx$ is

$$V(x) = \frac{1}{2}Kx^2 \tag{152}$$

Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}Kx^2\psi(x) = E\psi(x) \tag{153}$$

Change to dimenionless variables

$$y = \left(\frac{mK}{\hbar^2}\right)^{\frac{1}{4}} x = \alpha x; \qquad \epsilon = \frac{2E}{\hbar}\left(\frac{m}{K}\right)^{\frac{1}{2}} = \frac{2E}{\hbar\omega} = \frac{2E}{h\nu} \tag{154}$$

giving

$$\frac{d^2\psi}{dy^2} - y^2\psi(y) = -\epsilon\psi(y) \tag{155}$$

Complimentary solution

First solve simpler equation

$$\frac{d^2\psi}{dy^2} - y^2\psi(y) = 0 \tag{156}$$

(can think of this as equation as $|y| \to \infty$). Gives

$$\psi(y) = A\exp\left(-\frac{1}{2}y^2\right) + B\exp\left(\frac{1}{2}y^2\right) \tag{157}$$

Boundary conditions for a localised problem give $B = 0$ so that $\psi \to 0$ as $|y| \to \infty$. Assume full solution of form

$$\psi(y) = H(y)\exp\left(-\frac{1}{2}y^2\right)$$

$$\frac{d\psi}{dy} = \frac{dH}{dy}\exp\left(-\frac{1}{2}y^2\right) - y\psi$$

$$\frac{d^2\psi}{dy^2} = \frac{d^2H}{dy^2}\exp\left(-\frac{1}{2}y^2\right) - y\frac{dH}{dy}\exp\left(-\frac{1}{2}y^2\right) - \psi - y\frac{dH}{dy}\exp\left(-\frac{1}{2}y^2\right) + y^2\psi \tag{158}$$

which gives

$$\frac{d^2\psi}{dy^2} - y^2\psi(y) = -\epsilon\psi(y) = \exp\left(-\frac{1}{2}y^2\right)\left[\frac{d^2H}{dy^2} - 2y\frac{dH}{dy} - H\right] \tag{159}$$

so the equation to solve is

$$\frac{d^2H}{dy^2} - 2y\frac{dH}{dy} + (\epsilon - 1)H = 0 \tag{160}$$

This equation has no singular points. So can obtain two simple series solution about $y = 0$, these will have radius of convegence, $\rho = \infty$ (see 4.2). Also note that the equation is **even** so expect separate even and odd solutions

$$H(y) = \sum_{n=0}^{\infty} a_n y^n;$$

$$\frac{dH}{dy} = \sum_{n=0}^{\infty} n a_n y^{n-1};$$

$$\frac{d^2H}{dy^2} = \sum_{n=0}^{\infty} n(n-1) a_n y^{n-2} \tag{161}$$

so

$$\sum_{n=0}^{\infty} n(n-1) a_n y^{n-2} - 2\sum_{n=0}^{\infty} n a_n y^n + (\epsilon - 1)\sum_{n=0}^{\infty} a_n y^n = 0 \tag{162}$$

tidying this up and changing the dummy variable on the first sum by $n \rightarrow n+2$ gives

$$\sum_{n=-2}^{\infty} (n+1)(n+2) a_n y^n + \sum_{n=0}^{\infty} (\epsilon - 1 - 2n) a_n y^n = 0 \tag{163}$$

For this equation to be true for **all** values of y, the coefficient of each power of y must be **separately** equated to zero. This gives

$$2a_2 + (\epsilon - 1)a_0 = 0 \qquad \text{coef. of } y^0;$$

$$a_{j+2}(j+2)(j+1) - [\epsilon - 1 - 2j]a_j = 0 \qquad \text{coef. of } y^j. \tag{164}$$

giving a recurrence relation

$$a_{j+2} = \frac{2j - \epsilon + 1}{(j+1)(j+2)} a_j \qquad j = 0, 1, 2, \ldots \tag{165}$$

The series must **terminate** otherwise $H(y)$ and hence $\psi(x)$ go as $\exp(y^2/2)$, ie as the solution already rejected. If highest power of y in a solution is y^n, then a_{n+1} and a_{n+2} **must be zero**. This means

$$a_{n+2} = 0 = \frac{2n - \epsilon + 1}{(n+1)(n+2)} a_n \tag{166}$$

which gives

$$2n - \epsilon + 1 = 0 \tag{167}$$

or $\epsilon = 2n + 1$ as the physically allowed levels of the HO, which are

$$E = (n + \frac{1}{2})h\nu = (n + \frac{1}{2})\hbar\omega \qquad n = 0, 1, 2, \ldots \tag{168}$$

The polynomials $H(y)$ are called **Hermite Polynomials**, generally written $H_n(y)$. By convention they are written so that $a_n = 2^n$. They have recurrence relation

$$a_{j+2} = \frac{2(j-n)}{(j+1)(j+2)} a_j. \tag{169}$$

First few Hermite polynomials

$$H_0(y) = 1,$$

$$H_1(y) = 2y,$$

$$H_2(y) = 4y^2 - 2, \tag{170}$$

Normalisation constant:

$$N_n = \left(\frac{\alpha}{\pi^{\frac{1}{2}} 2^n n!}\right)^{\frac{1}{2}} \tag{171}$$

4.2 Special points

To know about the solubility of DE (133) it is necessary to analyse its structure. To do this it is convenient to re-write the equation by dividing through by $P(x)$ to give

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0\,, \tag{172}$$

where

$$p(x) = \frac{Q(x)}{P(x)}, \quad q(x) = \frac{R(x)}{P(x)}\,.$$

A DE of this form can have three types of points:

1. **Ordinary points**. x_o is an ordinary point if

   $$\lim_{x\to x_0}[p(x)] \quad \text{and} \quad \lim_{x\to x_0}[q(x)] \tag{173}$$

   are both finite. Most points are ordinary, eg all x in both classical and quantal HO problems. Legendre's equation (132) can be arrange into the form

   $$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{\ell(\ell+1)}{1-x^2}y = 0\,. \tag{174}$$

   It is ordinarly at $x_0 = 0$.

   If $p(x_0)$ and/or $q(x_0)$ are not ordinary, they are singular. For example $x_0 = \pm 1$ above or $\tan x$ is singular at $(2n+1)\frac{\pi}{2}$ for integer n.

   Simple series solutions of the type (134) can be used to find both solutions about an ordinary point.

2. **Regular singular points** are singularities such that

   $$p_0 = \lim_{x\to x_0}[(x-x_0)p(x)] \quad \text{and} \quad q_0 = \lim_{x\to x_0}[(x-x_0)^2 q(x)] \tag{175}$$

are finite. For example $x_0 = 1$ is a regular singular point in Legendre's equation (174).

Frobenius' method works about regular singular points. Furthermore, it can be shown (Fuch's theorem) that there exists at least one series solution about any regular singular point.

3. **Essentual singular points** are such that p_0 and/or q_0 are singular.

   Series solution methods cannot be used for essentual singular points.

From now on will always assume that $x_0 = 0$. If this is not the case it is easier to make a change of variable using $t = x - x_0$ than to work with an expansion about $x_0 \neq 0$.

For series solutions the **radius of convergence**, which is the largest value of x for which the series can be used, is is given by the singularity nearest to x_0 in the *complex plane*. Examples will be given below.

4.3 Indicial equations

Frobenius' method is based on using a generalisation of the series solution (134) used to solve for series about regular points. Assuming one is expanding about $x_0 = 0$, this takes the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+k} \tag{176}$$

with constraint that $a_0 \neq 0$. This equation has an extra x^k compared to expansion (134). k can take any value: it does not need to be integer, positive or even real. It is determined from something called the indicial equation.

To derive the indicial equation insert the expansion

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n\, x^{n+k}\,, \\ \frac{dy}{dx} &= \sum_{n=0}^{\infty} (n+k)\, a_n\, x^{n+k-1}\,, \\ \frac{d^2y}{dx^2} &= \sum_{n=0}^{\infty} (n+k)(n+k-1)\, a_n\, x^{n+k-2}\,. \end{aligned} \tag{177}$$

into DE (172) and obtain

$$\sum_{n=0}^{\infty} (n+k)(n+k-1)\, a_n\, x^{n+k-2} + p(x) \sum_{n=0}^{\infty} (n+k)\, a_n\, x^{n+k-1} + q(x) \sum_{n=0}^{\infty} a_n\, x^{n+k} = 0\,. \tag{178}$$

Multiply this equation through by x^{2-k}

$$\sum_{n=0}^{\infty}(n+k)(n+k-1)\,a_n\,x^n + p(x)\,\sum_{n=0}^{\infty}(n+k)\,a_n\,x^{n+1} + q(x)\,\sum_{n=0}^{\infty}a_n\,x^{n+2} = 0\,, \qquad (179)$$

then consider its value at the expansion point $x = 0$. At this point it is only necessary to consider terms with $n = 0$ since those with $n > 0$ are all zero at $x = 0$. This gives

$$k(k-1)\,a_0 + p(x)\,k\,a_0\,x + q(x)\,a_0\,x^2 = 0\,. \qquad (180)$$

Dividing through by a_0, since $a_0 \neq 0$, and remembering the definitions of p_0 and q_0 given by (175) one obtains

$$k(k-1) + p_0 k + q_0 = 0 \qquad (181)$$

which is a quadratic equation for k and is known as the indicial equation. I find this general form easiest to use but, as shown below, the indicial equation can be derived for each case.

If k_1 and k_2 are the two solutions of the indicial equation, then there are two possibilities:

1. If $k_1 - k_2 \neq$ an integer, then both solutions can be obtained in the form:

$$\begin{aligned} y_1(x) &= x^{k_1}[a_0 + \sum_{n=1}^{\infty} a_n\,x^n]\,, \\ y_2(x) &= x^{k_2}[a_0 + \sum_{n=1}^{\infty} b_n\,x^n]\,, \end{aligned} \qquad (182)$$

2. Otherwise, assuming $k_1 \leq k_2$, one solution has form of $y_2(x)$. Other may look like $y_1(x)$, but Fuch's theorem only guarantees that solution with k_2 will exist in this in form.

   One can find the second ("irregular") solution by letting $y(x) = y_2(x)\,v(x)$ and getting a simpler equation for $v(x)$. Often $v(x)$ has a nasty $\ell n(x)$ term in it. This is always the case if the indicial equation has equal roots, *i.e.* $k_1 = k_2$. This happens for Bessel's equation which one often comes across in problems with cylindrical symmetry.

Roots differing by an integer

Consider the equation

$$x(x-1)\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0\,.$$

Comparing this with the standard form gives $p(x) = \dfrac{3}{(x-1)}$ and $q(x) = \dfrac{1}{x(x-1)}$. Thus $x = 0$ and $x = 1$ are regular points of the differential equation and so we can expect to get at least one power series solution in x. At $x = 0$, get $k = 0$ and 1.

Inserting

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n\, x^{n+k}\,, \\ \frac{dy}{dx} &= \sum_{n=0}^{\infty} (n+k)\, a_n\, x^{n+k-1}\,, \\ \frac{d^2y}{dx^2} &= \sum_{n=0}^{\infty} (n+k)(n+k-1)\, a_n\, x^{n+k-2} \end{aligned}$$

into the differential equation,

$$\sum_{n=0}^{\infty} (n+k)(n+k-1)\, a_n\, (x^{n+k} - x^{n+k-1}) + \sum_{n=0}^{\infty} 3(n+k)\, a_n\, x^{n+k} + \sum_{n=0}^{\infty} a_n\, x^{n+k} = 0\,.$$

Hence

$$\sum_{n=0}^{\infty} a_n\, x^{n+k}\, [(n+k)(n+k-1) + 3(n+k) + 1] = \sum_{n=-1}^{\infty} (n+k+1)(n+k)\, a_{n+1}\, x^{n+k}\,.$$

Indicial equation comes from looking at the lowest power of x, given by $n = -1$ on the right hand side. Gives $k(k-1) = 0$, *i.e.* $k = 1$ or $k = 0$ as above.

Equating higher powers of x gives recurrence relation:

$$(n+k+1)^2\, a_n = (n+k+1)(n+k)\, a_{n+1}\,,$$

$$a_{n+1} = \left(\frac{n+k+1}{n+k}\right) a_n\,.$$

Recurrence relations allow us to evaluate all the higher coefficients from the first one. To fix the first term one has to remember that $a_0 \neq 0$. This term then acts as a scale constant for the whole solution, which must be determined from the boundary conditions.

Taking the index $k = 1$ and putting $a_0 = 1$, we get $a_1 = 2$, $a_2 = 3$ *etc.* The full solution is

$$y_1(x) = x(1 + 2x + 3x^2 + 4x^3 + \cdots) = \frac{x}{(1-x)^2}\,.$$

Note that this series converges for $\mid x \mid < 1$; the divergence at $x \geq 1$ is due to the singular point there.

On the other hand, when the index $k = 0$, we are in trouble because the recurrence relation is

$$a_{n+1} = \left(\frac{n+1}{n}\right) a_n \, .$$

If you try to calculate a_1 by putting $n = 0$ you see that the whole thing blows up. Hence there is not a second *series* solution at $x = 0$. Fuch's theorem only guaranteed that there would be one solution of this kind; the other solution is going to be nasty at $x = 0$.

4.4 Applying Frobenius's method

Consider the equation

$$2x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1+x)y = 0 \, . \tag{183}$$

For this $p(x) = \dfrac{-1}{2x}$ and $q(x) = \dfrac{1+x}{2x^2}$. This eq. has a regular singular point at $x = 0$ with $p_0 = \dfrac{-1}{2}$ and $q_0 = \dfrac{1}{2}$, giving an indicial eq.

$$k(k-1) - \frac{1}{2}k + \frac{1}{2} = 0 \, . \tag{184}$$

$$2k^2 - 3k + 1 = (2k-1)(k-1) = 0 \, . \tag{185}$$

Hence $k = \frac{1}{2}$ and 1.

Expanding as a series

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n \, x^{n+k} \, , \\ \frac{dy}{dx} &= \sum_{n=0}^{\infty} (n+k) \, a_n \, x^{n+k-1} \, , \\ \frac{d^2y}{dx^2} &= \sum_{n=0}^{\infty} (n+k)(n+k-1) \, a_n \, x^{n+k-2} \, . \end{aligned} \tag{186}$$

and substituting into DE (183) gives

$$\sum_{n=0}^{\infty} 2(n+k)(n+k-1) \, a_n \, x^{n+k} - \sum_{n=0}^{\infty} (n+k) \, a_n \, x^{n+k} + \sum_{n=0}^{\infty} a_n \, x^{n+k} + \sum_{n=0}^{\infty} a_n \, x^{n+k+1} = 0 \, . \tag{187}$$

For (187) to be satisfied for all values of x then the coefficient of each power of x must be zero. For x^k this gives

$$2k(k-1) - k + 1 = 2k^2 - 3k + 1 = 0 \, , \tag{188}$$

assuming $a_0 \neq 0$. This is the indical eq. derived above. Equating powers of x^{n+k} get

$$[2(k+n)(k+n-1) - (k+n) + 1]a_n + a_{n-1} = 0$$

$$a_n = \frac{-1}{2(k+n)^2 - 3(k+n) + 1} a_{n-1} = \frac{-1}{[2(k+n)-1][(k+n)-1]} a_{n-1}\,, \qquad n \geq 1\,. \tag{189}$$

Consider $k = 1$ and $k = \frac{1}{2}$ in turn.

$k = 1$

$$a_n = \frac{-1}{[2n+1]n} a_{n-1}\,, \qquad n \geq 1\,. \tag{190}$$

So

$$a_1 = -\frac{a_0}{3.1}\,,$$
$$a_2 = -\frac{a_1}{5.2} = \frac{a_0}{3.5(1.2)}\,,$$
$$a_3 = -\frac{a_2}{7.3} = \frac{a_0}{(3.5.7)(1.2.3)}\,,$$

etc. In general

$$a_n = \frac{(-1)^n}{[3.5.7\ldots(2n+1)]n!} a_0\,, \qquad n \geq 1\,. \tag{191}$$

Gives first general solution of (183), omiting constant multiplier a_0,

$$y_1(x) = x\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{[3.5.7\ldots(2n+1)]n!}\right]\,. \tag{192}$$

Use the ratio test to determine the radius of convergence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\frac{|x|}{(2n+3)(n+1)} = 0\,, \tag{193}$$

so series converges for all x. Can be seen as no poles in $p(x)$ and $q(x)$ except at expansion point $x = 0$.

$k = \frac{1}{2}$

Same method gives independent solution

$$y_2(x) = x^{\frac{1}{2}}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{[3.5.7\ldots(2n-1)]n!}\right]\,. \tag{194}$$

which is slso convergent for all x.

Now apply this method to commonly occuring physical problems, particularly those that arise from trying to solve Schrödinger's equation.

Laguerre's equation

In 2B22 the radial eq. of the H atom is written:

$$\frac{d^2F}{dr^2} - 2\kappa\frac{dF}{dr} + \left[\frac{2Z}{r} - \frac{\ell(\ell+1)}{r^2}\right] F = 0$$

where Z is the charge on the atom, ℓ the orbital angular momentum of the electron is an integer ≥ 0, κ its energy, and the range is $0 \leq r \leq \infty$. This equation is also known as Laguerres equation.

Analysing this eq. for $r \to 0$ shows:

$$p(r) = -2\kappa, \qquad p_0 = 0;$$

$$q(r) = \left[\frac{2Z}{r} - \frac{\ell(\ell+1)}{r^2}\right], \qquad q_0 = -\ell(\ell+1). \tag{195}$$

There are no other singular points so expanding about this point will give solutions with radius of convergence, $\rho = \infty$.

Indicial equation:

$$k(k-1) + kp_0 + q_0 = 0$$

$$k(k-1) - \ell(\ell+1) = 0$$

$$k^2 - k - \ell(\ell+1) = 0$$

$$(k+\ell)(k-(\ell+1)) = 0$$

$$k = -\ell, \ell+1$$

Solution with $k = -\ell$

$$F(r) = \sum_{n=0} a_n r^{n-\ell}, \qquad a_0 \neq 0$$

is unbounded (ie ∞) at $r = 0$, so unphysical.

Solution $k = \ell + 1$ gives

$$F(r) = \sum_{n=0} a_n r^{n+\ell+1}, \qquad a_0 \neq 0$$

rewrite as

$$F(r) = \sum_{j=\ell+1} a_j r^j, \qquad a_{\ell+1} \neq 0$$

$$\frac{dF}{dr} = \sum_j j a_j r^{j-1}$$

$$\frac{d^2F}{dr^2} = \sum_j j(j-1) a_j r^{j-2}$$

Substituting in

$$\sum_{j=\ell+1} j(j-1)a_j r^{j-2} - 2\kappa \sum_{j=\ell+1} j a_j r^{j-1} + 2Z \sum_{j=\ell+1} a_j r^{j-1} - \ell(\ell+1) \sum_{j=\ell+1} a_j r^{j-2} = 0$$

$$\sum_{j=\ell+1} [j(j-1) - \ell(\ell+1)]a_j r^{j-2} = \sum_{j=\ell+1} [2\kappa j - 2z]a_j r^{j-1}$$

$$\sum_{j=\ell} [j(j+1) - \ell(\ell+1)]a_{j+1} r^{j-1} = \sum_{j=\ell+1} [2\kappa j - 2Z]a_j r^{j-1}.$$

Equating powers of r^{j-1} gives

$$[j(j+1) - \ell(\ell+1)]a_{j+1} = [2\kappa j - 2z]a_j$$

and the recurrence relation

$$a_{j+1} = \frac{(2\kappa j - 2z)}{j(j+1) - \ell(\ell+1)} a_j$$

with $j > \ell$.

As $j \to \infty$

$$\frac{a_{j+1}}{a_j} \to \frac{2\kappa}{j+1}$$

which means that for large r, $F(r)$ behaves as $\exp(2\kappa r)$, remember

$$\exp(2\kappa r) = 1 + 2\kappa r + \frac{(2\kappa r)^2}{2!} + \ldots \frac{(2\kappa r)^n}{n!} \ldots$$

$\exp(2\kappa r)$ is not bounded so series must terminate.

Let n be highest term allowed, then $a_{n+1} = 0$

$$(2\kappa n - 2Z) = 0$$

$$\kappa = \frac{Z}{n} = (-2E)^{\frac{1}{2}}$$

which leads directly to the energy levels of the H atom

$$E = -\frac{Z^2}{2n^2}, \qquad n = 1, 2, 3, \ldots$$

with the condition $n > \ell$.

These solutions are known as Laguerre polynomials (terms can be written down using recurrence relation). These are orthogonal polynomials (like Hermite and Legendre). In fact there are a set for each value of ℓ.